Stochastic Compounding and Uncertain Valuation

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Abstract

Exploring long-term implications of valuation leads us to recover and use a distorted probability measure that reflects the long-term implications for risk pricing. This measure is typically distinct from the physical and the risk neutral measures that are well known in mathematical finance. We apply a generalized version of Perron-Frobenius theory to construct this probability measure and present several applications. We employ Perron-Frobenius methods to i) explore the observational implications of risk adjustments and investor beliefs as reflected in asset market data; ii) catalog alternative forms of misspecification of parametric valuation models; and iii) characterize how long-term components of growth-rate risk impact investor preferences implied by Kreps-Porteus style utility recursions.

1 Introduction

We explore ways to understand how compounding stochastic growth and discounting alters valuation. Our aim is to provide characterizations that allow for nonlinear stochastic

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specifications outside the realm of log-normal models or approximations often used in asset pricing models. The methods we describe allow for alternative components of valuation to become prominent over multiple investment horizons. We consider three related substantive problems.

First, we explore how compounding interacts with the assignment of risk prices. Here we use the the mathematically convenient construct of a stochastic discount factor process. Stochastic discount factors represent market valuation of risky cash flows. They are stochastic in order to adjust for risk as they discount the future. Multi-period risk adjustments, captured formally by “multiplicative” stochastic discount factors, reflect the impact of compounding single-period risk adjustments. Stochastic discount factors are only well defined relative to well specified probability distribution. By studying valuation in revealing ways, we are lead to address some challenges in identifying the role of investor beliefs versus stochastic discount factors. As is well known, when markets are complete and market prices are observed by an econometrician, for a given probability measure there is a unique stochastic discount factor process that is revealed from financial market data. Ross (2013) extends this claim to argue that the probability measure itself can also be recovered. By studying a class of stochastic economies with growth, we show that the recovery approach of Ross (2013) typically does not recover the underlying probability but instead the a long-term counterpart to a forward measure. See Section 3.

Second, we explore misspecification of parametric models of valuation. It is common to identify stochastic discount factor processes through parametric restrictions. Parametric models, while tractable, are typically misspecified. By studying how the consequences of the misspecification are related to the payoff or investment horizon, we motivate and catalog different forms of misspecification. For instance, economic fundamentals as specified in a parametric model could dominate over longer investment horizons. Thus econometric identification should reflect this possibility. Alternatively, statistically small forms of model misspecification might become more evident over longer time periods if the source is the misspecification of the underlying stochastic evolution as perceived by investors. See Section 4.

Third, recursive specifications of preferences of type first suggest by Kreps and Porteus (1978) are known to have nontrivial implications for economies with stochastic growth and volatility. By looking at long-term stochastic characterizations of consumption, we obtain tractable mathematical characterizations of the limiting specifications and in turn general characterizations of when solutions exist for the for the infinite horizon version of Kreps
and Porteus (1978) preferences. In effect we isolate long-term contributions to the risk-adjustments embedded in recursive utility. For specifications in which the impact of the future in the continuation values is particularly prominent, we show what aspects of the riskiness of the consumption process dominate valuation from the perspective of investor preferences. See Section 5.

We study these three problems using a common analytical approach. This approach starts by characterizing the limiting impact of compounding in a stochastic environment. It applies a generalized version of Perron-Frobenius theory for Markov processes and has much in common with large deviation theory as developed by Donsker and Varadhan (1976). In Section 2 we describe the stochastic environment that underlies our analysis. We show how to use Perron-Frobenius theory to identify a state invariant growth or discount rate and an associated martingale. This martingale induces an alternative probability measure that helps to reveal the long-term contributions to growth and valuation. Thus Section 2 lays out the mathematical tools that facilitate our study in subsequent sections.

2 A Factorization Result

We begin with a Markov representation of stochastic growth or discounting. This representation is convenient for analyzing the impact of compounding in a stochastic environment. Here we use a discrete-time formulation as in Hansen and Scheinkman (2012a). Continuous-time counterparts have been developed in Hansen and Scheinkman (2009) and Hansen (2012). We apply a general version of Perron-Frobenius theory in making this investigation. Recall that in the case of a matrix $A$, such that $A^k$ has all strictly probative entries for some positive integer $k$, Perron-Frobenius theory implies that $A$ has a positive eigenvalue that is associated with a positive eigenvector. This positive eigenvalue dominates in absolute value all other eigenvalues of $A$ and thus dominates the exponential growth rate of $A^k$ as $k \to \infty$. Perron-Frobenius theory generalizes to a class of non-negative linear operators. By applying this approach, we isolate components of growth and valuation that become much more prominent over longer horizons.

We start with a joint Markov process $(X, Y)$:

**Assumption 2.1.** The joint Markov process $(X, Y)$ is stationary.

We could weaken this by imposing some form of stochastic stability, while not initializing the process using the stationary distribution.
Assumption 2.2. The joint distribution of \((X_{t+1}, Y_{t+1})\) conditioned on \((X_t, Y_t)\) depends only on \(X_t\).

In light of this restriction, we may view \(X\) alone as a Markov process and \(Y\) does not “cause” \(X\) in the sense of Granger (1969). Moreover, the process \(Y\) can be viewed as an independent sequence conditioned on the entire process \(X\) where the conditional distribution of \(Y_t\) depends only on \(X_t\) and \(X_{t-1}\). The role of \(Y\) is a device to introduce an additional source of randomness, but it allows us to focus on the intertemporal impact using a smaller state vector process \(X\).

Construct a process \(M\) such that

\[
\log M_{t+1} - \log M_t = \kappa(X_{t+1}, Y_{t+1}, X_t).
\]

This process has stationary increments. As a result of this construction, the process \(M\) will grow or decay stochastically over time, and it is convenient to have methods to characterize this stochastic evolution. Examples of \(M\) include stochastic growth processes. These processes could be macro times series expressed in levels that inherit stochastic growth along some balanced growth path or stochastic discount factor processes used to represent equilibrium asset values. In what follows, we will also have use for

\[
\bar{\kappa}(X_{t+1}, X_t) = \log E \left( \exp \left[ \kappa(X_{t+1}, Y_{t+1}, X_t) | X_{t+1}, X_t \right] \right),
\]

The following example illustrates this construction:

Example 2.3. Consider a dynamic mixture of normals model. Suppose that \(X\) evolves as an \(n\)-state Markov chain and that \(Y\) is an iid sequence of standard normally distributed random vectors in \(\mathbb{R}^m\). Let the realized values of the state vector \(X_t\) be the coordinate vectors \(u_i\) for \(i = 1, 2, \ldots, n\) where \(u_i\) is a vector of zeros except in entry \(i\) where it equals 1. Suppose \(\bar{\beta} \in \mathbb{R}^n\) and \(\bar{\alpha}\) is a \(m \times n\) matrix. Let

\[
\log M_{t+1} - \log M_t = \kappa(X_{t+1}, Y_{t+1}, X_t) := \bar{\beta} \cdot X_t + (X_{t+1})' \bar{\alpha}' Y_{t+1}.
\]

The current state \(X_t\) affects the growth in \(M\) in two ways. First, if the state at time \(t\) is \(i\), the average growth of \((\log) M\) between \(t\) and \(t+1\) is \(\bar{\beta}_i\). In addition, the state at \(t\) determines the distribution of \(X_{t+1}\) and thus the variance of the growth of \((\log) M\). In a more elaborate model, the state could also affect the distribution of \(Y_{t+1}\), but in this example we assume that \(Y\) is iid.
Since \( \kappa(X_{t+1}, Y_{t+1}, X_t) \), conditioned on \( X_t \) and \( X_{t+1} \), is normally distributed,

\[
\bar{\kappa}(X_{t+1}, X_t) = \bar{\beta} \cdot X_t + \frac{1}{2} (X_{t+1})' \bar{\alpha}' \bar{\alpha} X_{t+1}.
\]

Next we develop the discrete-time counterpart to an approach from Hansen and Scheinkman (2009) and Hansen (2012). This approach extends a lognormal formulation in Hansen et al. (2008). While log-normal specifications are commonly used because of their convenience they limit the channels by which locally subtle statistical components can become prominent. We use the process \( M \) to construct one-period operators and then explore the impact of applying these operators in succession multiple times. This sequential application reflects the impact of compounding. The first operator \( \mathbb{M} \) maps functions \( g(x, y) \) into functions of the state variable \( x \), \( \mathbb{M}g \) via:

\[
[\mathbb{M}g](x) = E \left( \exp[\kappa(X_{t+1}, Y_{t+1}, X_t)]g(X_{t+1}, Y_{t+1}) \big| X_t = x \right).
\]

Notice that the stationarity of \( (X, Y) \) makes the right hand side independent of \( t \). The second operator \( \overline{\mathbb{M}} \) maps functions \( f(x) \) of the state variable \( x \) into functions of the state variable \( x \), \( \overline{\mathbb{M}}f \) via:

\[
[\overline{\mathbb{M}}f](x) = E \left( \exp[\bar{\kappa}(X_{t+1}, X_t)]f(X_{t+1}) \big| X_t = x \right)
\]

The function \( g(x, y) \) may be independent of \( y \), that is \( g(x, y) = f(x) \). In this case the two operators are consistent. The Law of Iterated Expectations insures that:

\[
[\mathbb{M}f](x) = E \left( \exp[\kappa(X_{t+1}, Y_{t+1}, X_t)] f(X_{t+1}) \big| X_t = x \right)
= E \left( \exp[\bar{\kappa}(X_{t+1}, X_t)] f(X_{t+1}) \big| X_t = x \right)
= [\overline{\mathbb{M}}f](x)
\]

Thus \( \overline{\mathbb{M}} \) and \( \mathbb{M} \) agree when they are both well defined. For now we will be vague about the collection of functions \( g \) or \( f \) that are in the domain of the operators \( \mathbb{M} \) and \( \overline{\mathbb{M}} \).

Since \( \mathbb{M}g \) is only a function of the state \( x \), for any \( j \geq 1 \),

\[
[\mathbb{M}^j g](x) = \overline{\mathbb{M}}^{j-1}[\mathbb{M}g](x).
\]

This relation shows that, when we look across multiple horizons, we can concentrate our attention on the operator \( \overline{\mathbb{M}} \) and its iterates featuring the \( X \) dependence.
In addition, again applying the Law to Iterated Expectations we obtain:

\[
[M^j g](x) = E \left[ \left( \frac{M_{t+j}}{M_t} \right) g(X_{t+j}, Y_{t+j}) | X_t = x \right].
\]

Finally, since \( M \) is a positive process, the operators \( M \) and \( \overline{M} \) map positive functions into positive functions.

**Example 2.4.** It is straightforward to compute the operator \( \overline{M} \) in Example 2.3. Since \( X \) is an \( n \)-state Markov chain, functions of \( x \) can be identified with vectors in \( \mathbb{R}^n \), and the linear operator \( \overline{M} \) can be identified with its matrix representation \( A = [a_{ij}] \). Applying \( \overline{M} \) to \( u_j \) amounts to computing \( Au_j \) and reveals the \( j \)-th column of \( A \). Let \( P = [p_{ij}] \) be the transition matrix for \( X \). Then it is easy to show that:

\[
a_{ij} = p_{ij} \xi_{ij}
\]

where

\[
\xi_{ij} = \exp \left[ \beta \cdot u_i + \frac{1}{2} (u_j)' \bar{\alpha} \bar{\alpha}' u_j \right].
\]

Thus the probabilities are adjusted by growth or decay factors, \( \xi_{ij} \), but all entries remain positive. If \( A^{j_0} \) has strictly positive entries for some \( j_0 \), then the Perron-Frobenius Theorem states that there exists a unique (up to scale) eigenvector \( e \) satisfying:

\[
Ae = \exp(\eta)e
\]

with strictly positive entries. The eigenvalue associated with this positive eigenvector is positive and has the largest magnitude among all of the eigenvalues. As a consequence, this eigenvalue and associated eigenvector dominate as we apply \( \overline{M} \) \( j \) times, for \( j \) large.

For general state spaces we consider the analogous Perron-Frobenius problem, namely, find a strictly positive function \( e(x) \) such that there exists an \( \eta \) with

\[
[M e](x) = [\overline{M} e](x) = \exp(\eta)e(x)
\]

Existence and uniqueness are more complicated in the case of general state spaces. Hansen and Scheinkman (2009) presents sufficient conditions for the existence of a solution, but even in examples commonly used in applied work, multiple (scaled) positive solutions are a possibility. See Hansen and Scheinkman (2009) and Hansen (2012) for such examples.
However, when we have a solution of the Perron-Frobenius problem, we follow Hansen and Scheinkman (2009), and define a process \( \tilde{M} \) that satisfies:

\[
\frac{\tilde{M}_t}{M_0} = \exp(-\eta t) \left( \frac{M_t}{M_0} \right) \frac{e(X_t)}{e(X_0)}. \tag{2}
\]

While this leaves open the question of how to initialize \( \tilde{M}_0 \), the initialization is inconsequential to much of what follows as long as \( \tilde{M}_0 \) is strictly positive. Notice that by construction:

\[
E\left( \frac{\tilde{M}_{t+1}}{\tilde{M}_t} \middle| F_t \right) = E\left( \frac{\tilde{M}_{t+1}}{\tilde{M}_t} \middle| X_t \right) = 1
\]

implying that \( \tilde{M} \) is a positive \( \{F_t : t = 0, 1, \ldots\} \)-martingale, when \( F_t \) is the sigma algebra of events generated by \( X_0, X_1, \ldots, X_t \) and \( Y_1, Y_2, \ldots, Y_t \). Moreover,

\[
\log \tilde{M}_{t+1} - \log \tilde{M}_t = -\eta + \kappa(X_{t+1}, Y_{t+1}, X_t) + \log e(X_{t+1}) - \log e(X_t)
\]

\[
= \tilde{\kappa}(X_{t+1}, Y_{t+1}, X_t).
\]

Thus the process \( \tilde{M} \) has the same mathematical structure and the original process \( M \). In logarithms, this constructed process has stationary increments represented as a function of \( X_{t+1}, Y_{t+1} \) and \( X_t \).

An outcome of this construction is the factorization that helps us to understand how compounding works in this Markov environment. Inverting (2):

\[
\frac{M_t}{M_0} = \exp(\eta t) \left( \frac{\tilde{M}_t}{\tilde{M}_0} \right) \left[ \frac{\tilde{e}(X_t)}{\tilde{e}(X_0)} \right] \tag{3}
\]

where \( \tilde{e} = \frac{1}{e} \). The first term in this factorization determines a deterministic growth (when \( \eta > 0 \)) or decay (when \( \eta < 0 \)) of the \( M \) process. The last term is a fixed function of the (stationary) process \( X_t \). Except when the martingale is degenerate, the middle term is a stochastic contribution to compounding. The positive martingale term can be used for a “change of measure”. Specifically,

\[
\tilde{M} f(x) = E \left( \exp \left[ \tilde{\kappa}(X_{t+1}, Y_{t+1}, X_t) \right] f(X_{t+1}) \middle| X_t = x \right)
\]

defines the conditional expectation operator implied by the transition distribution for a
Markov process. The term \( \text{exp}\left[\tilde{\kappa}(X_{t+1}, Y_{t+1}, X_t)\right] \) is the relative density or Radon-Nykodym derivative or a new transition distribution relative to the original transition distribution. In what follows we suppose that the following stochastic stability condition is satisfied

\[
\lim_{j \to \infty} \tilde{M}_j f(x) = \int f(x)\tilde{Q}(dx) \quad (4)
\]

where \( \tilde{Q} \) is a stationary distribution for the transformed Markov process. When there are distinct solutions to the Perron-Frobenius problem, at most one solution will satisfy this stochastic stability requirement (see Hansen and Scheinkman (2009)). This stochastic stability is what permits the change of probability measure to be valuable for characterizing long-horizon limits. The change-of-measure captures the long-term impact of the stochastic component to compounding.

In fact, we may write

\[
\text{exp}\left(-\eta_j\right)M_j f(x) = e(x)E\left[\frac{\tilde{M}_j}{M_0} f(X_t)\tilde{e}(X_t)|X_0 = x\right] = e(x)\tilde{M}_j[f\tilde{e}](x)
\]

The stability condition (4) allows us to interpret \( \eta \) as the asymptotic rate of growth.

We next explore some applications of this factorization.

3 Stochastic discount factors

As in Hansen and Richard (1987), we use stochastic discount factors both to discount the future and adjust for risk. They serve as “kernels” for pricing operators that assign current period prices to future payoffs. The stochastic discount factor process captures both an interest rate adjustment and a risk-neutral measure adjustment. Standard price-theoretic reasoning connects these discount factors to intertemporal marginal rates of substitution of investors. Let \( S \) be such a process implied by an equilibrium asset pricing model, a Markov model that is consistent with balanced stochastic growth. Specifically, we use \( \frac{S_t}{S_0} \) as the stochastic discount factor for assigning date zero prices to payoffs at date \( t \).

We presume that \( S \) has the mathematical structure described in the previous section for a generic process \( M \); and we use the stochastic discount factor process \( S \) to construct a family of valuation operators indexed by the investment horizon. We use this martingale from Section 2 in conjunction with stochastic discount factors to show how this alternative
probability measure can be revealed by financial market data. When the martingale that induces the alternative probability measure is a constant, the alternative measure is identical to the actual probability measure, and thus we obtain the “recovery” result of Ross (2013). In an equilibrium model, however, the martingale of interest is unlikely to be a constant unless consumption is stationary (up to a deterministic rate of growth) as in examples discussed in Ross (2013). In the stochastic environment used in this paper, the probability measure recuperated using Perron-Frobenius is typically not the one-period transition distribution, but it is an altered measure that can be used to characterize value implications for long investment horizons.

Let $S$ be a valuation operator that assigns date $t$ prices to date $t+1$ payoffs that are functions of a Markov state. Specifically, we allow the payoffs to be functions of both $X_{t+1}$ and $Y_{t+1}$. To infer the prices of multi-period payoffs we iterate on the one-period valuation operator. Given a characterization of the pricing of these “primitive” payoffs, we can extend the valuation operator to an even richer collection of asset payoffs with more complicated forms of history dependence.

We apply factorization (3) to the stochastic factor process:

$$\frac{S_t}{S_0} = \exp(\eta t) \left( \frac{\tilde{S}_t}{\tilde{S}_0} \right) \left[ \tilde{e}(X_t) \right].$$

(5)

where $\tilde{e} = \frac{1}{e}$ and $e, \eta$ are solutions to the Perron-Frobenius problem associated with $S$

$$Se = \tilde{S}e = \exp(\eta)e.$$ 

Since $S$ discounts, we expect $\eta$ to be negative. Additionally, $-\eta$ is the limiting interest rate on a long-term discount bond provided that $\tilde{e}$ has a finite expectation under the change of measure induced by the positive martingale $\tilde{S}$.

Alvarez and Jermann (2005) use factorization (5) to argue for the importance of permanent shocks as operating through the martingale component $\tilde{S}$. Specifically, they interpret the multiplicative factorization analogously to an additive counterpart obtained, say by taking logarithms of $S$. The additive martingale extraction is familiar from time series analysis.
and empirical macroeconomics as a device to identify permanent shocks. The martingale component in a multiplicative factorization is positive and has “unusual” sample path properties. It converges almost surely, and for many example economies with stochastic growth it converges to zero. Hansen and Scheinkman (2009) and Hansen (2012) instead use the martingale component as a change of measure and show in what sense this change of measure has permanent consequences for pricing. As emphasized by Hansen (2012), if logS has a nondegenerate martingale component then so does S, and conversely. This relation gives a different but less direct way to motivate the analysis of permanent shocks in Alvarez and Jermann (2005). While there is a tight connection between the multiplicative martingale component of S and the additive martingale component of log S for log-normal specifications, in general there is no simple relation. In what follows we discuss further the implied change in probability distribution associated with the martingale component.

As we argued in Section 2, we may use $\tilde{S}$ to define a distorted conditional expectations operator as featured in Hansen and Scheinkman (2009) and Hansen (2012):

$$\tilde{E}[g(X_{t+1}, y_{t+1})|X_t = x] = E\left[\left(\frac{\tilde{S}_{t+1}}{S_t}\right) g(X_{t+1}, Y_{t+1})|X_t = x\right] = \exp(-\eta)\tilde{e}(x)[Sge](x) (6)$$

where as before, $\tilde{e} = \frac{1}{e}$. When markets are (dynamically) complete there is a unique operator $S$ compatible with the observed asset-prices and the absence of arbitrage. If a researcher has at his disposal the relevant information on asset prices, he can infer the operator S, and the (relevant) associated Perron-Frebenius eigenfunction e and eigenvalue $\exp(\eta)$. Alternatively, he might have a more limited amount of asset market data and instead infer $\tilde{S}$ by parameterizing an underling economic model. Thus right-hand side of (6) sometimes can be identified in a formal econometric sense, revealing the distorted expectation operator on the left-hand side. The recovered transition distribution is not the actual one-period transition distribution. Instead it is an altered measure that provides a convenient way to characterize value implications for long investment horizons. It can be viewed as the limiting analog of a

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3The “left over” part in additive decomposition is correlated with the permanent part, making it hard to use directly as a statistical decomposition. This correlation has led many researchers to identify “transitory shocks” as those that are uncorrelated with the martingale increment. Similarly, the Perron-Frobenius eigenfunction is correlated with with the martingale component, making it difficult to interpret (5) directly as a decomposition.

4In contrast, the martingale from an additive extraction obeys a central limit theorem when appropriately scaled.
forward measure used sometimes in mathematical finance.

**Example 3.1.** It is revealing to consider the special case of a finite-state Markov chain for $X$. For the time being, we abstract from the role of $Y$ and consequently restrict $S = \mathbb{S}$. The conditional expectation for such a process can be represented as a $n \times n$ matrix of transition probabilities, $P = [p_{ij}]$, and functions of the Markov state can be represented as vectors $f$ where entry $i$ is the value that the function takes in state $i$. Thus as in Example 2.3 there is also a matrix depiction $A = [a_{ij}]$ of the operator $S$ with $n^2$ entries, and applying $S$ to the $i^{th}$ coordinate vector (a vector with all zeros except in entry $i$) reveals the $i^{th}$ column of the matrix used to represent the operator $S$. In matrix terms, we solve

$$Ae = \exp(\eta)e.$$

It is of interest to consider the “inverse” of relation (6):

$$[S(f)](x) = \exp(\eta)\tilde{E}[f(X_{t+1})\tilde{c}(X_{t+1})|X_t = x]e(x).$$

(7)

Perron-Frobenius theory suggests representing $\mathbb{S}$ by the expression on the right-hand side of (7). Using the matrix representation $A$, equation (7) becomes:

$$a_{ij} = \exp(\eta)\tilde{p}_{ij} \left( \frac{e_i}{e_j} \right)$$

(8)

where the distorted transition matrix $\tilde{P} = [\tilde{p}_{ij}]$. Restricting the $\tilde{p}_{ij}$ to be transition probabilities in (8) guarantees that if $\tilde{P}$ and the vector $e$ solve the $n^2$ equations (8) then

$$(Ae)_i = \sum_j a_{ij}e_j = \exp(\eta)\sum_j \tilde{p}_{ij}e_i = \exp(\eta)e_i,$$

and hence $e$ is indeed an eigenvector of $A$ associated with eigenvalue $\exp(\eta)$. The transition matrix, $\tilde{P}$ has $(n^2 - n)$ free parameters. The $n$ dimensional eigenvector $e$ with positive entries is only identified up to scale and hence depends on $n - 1$ free parameters and the eigenvalue $\exp(\eta)$ gives one more free parameter. Thus, for this example we may think of (7) or equivalently (8) as providing $n^2$ equations with input $A$ to be used in identifying $n^2$ free parameters of $\tilde{P}$, $e$ and $\eta$. While Perron-Frobenius approach recovers a transition matrix, it is not the transition matrix for the underlying Markov chain.

If $\tilde{S}$, the martingale component of $S$, is constant over time, the distorted and actual
probabilities are identical and one obtains the recovery result of Ross (2013), who also uses Perron-Frobenius theory in his construction. This is a remarkable result because it allows for the recovery of the transition distribution of the Markov state and thus permits investor beliefs to be subjective. When marginal utility of an investor is a time-invariant function of the Markov state, we may interpret \( \delta = -\eta \) as a subjective discount rate and \( \tilde{c} \) as the marginal utility of consumption expressed as a function of the Markov state.

To justify the recovery of the actual (as opposed to distorted) transition density, we had to restrict the martingale component to be constant over time. Alvarez and Jermann (2005) argue why it is important empirically to allow for the martingale component in a stochastic discount factorization (5). In an equilibrium model, a specification where the martingale component is constant is unlikely to hold unless consumption is stationary, an assumption that typically is not made in the macro-asset pricing literature.\(^5\) In general, the application of Perron-Frobenius theory to one-period valuation recovers a transition distribution, but one that characterizes long-term valuation. This transition distribution will differ from the actual conditional distribution when there is a martingale component to the stochastic discount factor.

**Example 3.2.** Consider the dynamic normal mixture model from Example 2.3. In that example it is assumed that

\[
\log S_{t+1} - \log S_t = \bar{\beta} \cdot X_t + (X_{t+1})'\bar{\alpha}'Y_{t+1}
\]

This dynamics for \( S \) can be justified from more primitive assumptions. For instance, if the representative consumer has log utility and a subjective discount rate \( \delta \), the dynamics for \( S \) follow, provided we postulate that the consumption process \( C \) satisfies:

\[
\log C_{t+1} - \log C_t = \tilde{\beta} \cdot X_t + (X_{t+1})'\tilde{\alpha}'Y_{t+1}
\]

where

\[
\tilde{\beta} = -\bar{\beta} - \exp(-\delta)1_n
\]

\[
\tilde{\alpha} = -\bar{\alpha}
\]

and \( 1_n \) is an \( n \)-dimensional vector of ones. In equation (9), the expected change in \( \log C \)

\(^5\)As Alvarez and Jermann (2005) point out, even if consumption is stationary, the more general recursive utility model could imply a martingale component to the stochastic discount factor process.
between \( t \) and \( t + 1 \) is given by \( \tilde{\beta} \cdot u_i \) (the \( i^{th} \) component of \( \tilde{\beta} \)) when state \( i \) is realized at date \( t \). The variance of the change in \( \log C \) is \( (u_j)'\tilde{\alpha}'\tilde{\alpha}u_j \) (the \( j^{th} \) diagonal entry of \( \tilde{\alpha}'\tilde{\alpha} \)) conditioned on state \( j \) being realized at date \( t + 1 \). The vector \( \tilde{\beta} \) and matrix \( \tilde{\alpha} \) have analogous interpretations for the process \( \log S \).

We use this example to better illustrate the link between \( P \) and \( \tilde{P} \). We presume that state vector \((X, Y)\) is observable to investors and consider the recovery of the transition matrix for the state vector \( x \). For this example, equations (1) and (8) show that the recovered transition matrix implied by (6) is \( \tilde{P} \) where:

\[
\tilde{p}_{ij} = \exp(-\eta)p_{ij}\xi_{ij} \left( \frac{e_j}{e_i} \right).
\]

Direct verification shows that \( \tilde{P} \) is indeed a transition matrix. This transition matrix agrees with \( P \) when

\[
\exp(-\eta)\xi_{ij} \left( \frac{e_j}{e_i} \right) = 1
\]

for all \((i, j)\), but this will not be true in general. Equation (10) implies that

\[
\xi_{ij} = \exp(\eta) \left( \frac{e_i}{e_j} \right).
\]

The \( \xi_{ij} \)'s depend on \( 2n \) parameters, the \( n \) means of \( \log S_{t+1} - \log S_t \) and the \( n \) variances of \( \log S_{t+1} - \log S_t \) conditioned on \( X_t \) and \( X_{t+1} \). Thus there are \( 2n \) parameters that underly the \( \xi_{ij} \)'s on the left-hand side of this equation, but only \( n \) free parameters that we can vary on the right-hand side (where \( e \) only matters up to a free scale parameter).\(^6\)

To make more transparent how restrictive is the requirement in equation (11), suppose that \( n = 2 \) and \( Y \) is univariate. Then

\[
\xi_{ij} = \exp \left[ \tilde{\beta}_i + \frac{1}{2}(\tilde{\alpha}_j)^2 \right].
\]

If \( p_{ii} = \tilde{p}_{ii} \), then \( \xi_{ii} = \exp(\eta) \) and thus \( \tilde{\beta}_1 + \frac{1}{2}(\tilde{\alpha}_1)^2 = \tilde{\beta}_2 + \frac{1}{2}(\tilde{\alpha}_2)^2 \).

Appendix A describes an alternative recovery strategy, but also argues why this approach will not work under more general circumstances.\(^7\)

\(^6\)In this example the \( \xi_{ij} \)'s depend on \( 2n \) parameters but one can construct examples in which there are \( n^2 \) free parameters.

\(^7\)One might argue that we should expand our search for eigenfunctions to a broader set of functions. For instance, we might add \( S_t \) to the state vector along with \( X_t \) and \( Y_t \) even though \( \log S_{t+1} - \log S_t \) is constructed
A reader might be concerned that our focus on the finite state case is too special. What follows is an example based on an underlying Markov diffusion specification except that we allow for more shocks to be priced than the underlying state variable.

**Example 3.3.** Suppose that $X$ is a Markov diffusion:

$$dX_t = \mu(X_t)dt + \Lambda(X_t)dW_t$$

and

$$d\log S_t = \beta(X_t)dt + \alpha(X_t) \cdot dW_t.$$ 

The counterpart to $Y_{t+1}$ in this example is any component of the $dW_t$ increment that is not needed for the evolution of $X$. To construct the counterpart to $S$ in continuous time we consider the operator $\mathbb{G}$ that associates to each function $f$ the “expected time derivative” of $\frac{S_t}{S_0} f(X_t)$ at $t = 0$, that is the drift of $\frac{S_t}{S_0} f(X_t)$ at $t = 0$.

$$[\mathbb{G}(f)](x) = \left[ \beta(x) + \frac{1}{2} |\alpha(x)|^2 \right] f(x) + [\mu(X_t)' + \alpha(x) \Lambda(x)] \frac{\partial f(x)}{\partial x}$$

$$+ \frac{1}{2} \text{trace} \left[ \Lambda(x) \Lambda(x)' \frac{\partial^2 f(x)}{\partial x \partial x'} \right]$$

In this expression, $-\beta(x) - \frac{1}{2} |\alpha(x)|^2$ is the instantaneous interest rate and $-\alpha$ is the vector of risk prices that capture the expected return reward needed as compensation for exposure to $dW_t$. To apply Ito’s formula to compute the drift, we may restrict $f$ to be twice continuously differentiable; but we typically have to explore a larger collection of functions in order to solve eigenvalue problem:

$$\mathbb{G}e(x) = \eta e(x)$$ (12)

for a positive function $e$. See Hansen and Scheinkman (2009) for a more formal analysis.

---

from $X_{t+1}$, $Y_{t+1}$ and $X_t$. Since $S$ is typically not stationary, this would require relaxing some our maintained assumptions and would require that we dispense with the selection rule in Hansen and Scheinkman (2009). A potential gain to the approach is that we might view $\frac{1}{S_t}$ as a Perron-Frobenius eigenfunction associated with the unit eigenvalue based on the observation that $\frac{1}{S_t}$ is strictly positive. One could modify the selection rule of Hansen and Scheinkman (2009) requiring only the stochastic stability of $X$. This, however, would allow potentially for both candidate solutions. Alternatively, we could identify the eigenfunction of interest by “zeroing out” the dependence on $X_t$ and $Y_t$. If this is our selection criterion, we would choose $\frac{1}{S_t}$ but in this case the use of Perron-Frobenius theory adds no information concerning the dynamics of $X$.

---

8This operator is called the “generator for the pricing semigroup” in Hansen and Scheinkman (2009).
The diffusion dynamics implied by the Perron-Frobenius adjustment is:

\[ dX_t = \left[ \mu(X_t)' + \alpha(x)'\Lambda(x)' + \left[ \frac{\partial \log e(x)}{\partial x} \right]'\Lambda(x)\Lambda(x)' \right] dt + \Lambda(X_t)d\tilde{W}_t \]

where \( \tilde{W} \) is a Brownian motion under the change of measure. See Appendix B. The version of equation (6) for continuous time allows us to identify the distorted dynamics for \( X \). Thus in order for this procedure to identify the original dynamics it is necessary that

\[ \alpha(x) = -\Lambda(x)' \left[ \frac{\partial \log e(x)}{\partial x} \right]. \]

As is shown in Appendix B, when this restriction is satisfied we may write:

\[ \frac{S_t}{S_0} = \exp(\eta t) \frac{e(X_0)}{e(X_t)}, \]

implying that \( S \) has a constant martingale component.

4 Model misspecification

In this section we suggest an approach to consider transient model misspecifications whereby the candidate economic model of a stochastic discount factor is restricted to give the correct limiting risk prices. We contrast this approach to models of belief distortions that by design target the martingale components of stochastic discount factors. When analyzing models with belief distortions, Perron-Frobenius theory gives a way to assess how large the distortions are from a statistical perspective.

As the examples in section 3 suggest, Perron-Frobenius theory recovers an interesting probability distribution but typically one that is distorted. To proceed under stochastic growth, we must bring to bear additional information. Otherwise we would be left with a rather substantial identification problem. Following the literature on rational expectations econometrics, we can appeal to “cross equation restrictions” if the Markov state vector process \( X \) is observable. In what follows we sketch two related approaches for evaluating parametric stochastic discount factor models of valuation. These approaches are distinct but complementary to the nonparametric approach proposed by Ait-Sahalia and Lo (2000).\(^9\)

\(^9\)Ait-Sahalia and Lo (2000) show how to recover a stochastic discount factor nonparametrically over a given investment horizon using option prices to extract a forward measure and forming the ratio of the
4.1 Transient model misspecification

Building from Bansal and Lehmann (1997) and subsequent research, in Hansen (2012) we relate factorization (3) to a stochastic discount factor representation

\[ \frac{S_t}{S_0} = \frac{S^*_t}{S^*_0} \left[ \frac{h(X_t)}{h(X_0)} \right] \]  

(13)

where \( \frac{S^*_t}{S^*_0} \) is the stochastic discount factor in representative consumer power utility model and \( h \) modifies the investor preferences to include possibly internal habit persistence, external habit persistence or limiting versions of recursive utility. In the case of internal and external habit persistence models, these modifications may entail an endogenous state variable constructed on the basis of current and past consumptions. Introducing this function \( h \) modifies the Perron-Frobenius problem by leaving the eigenvalue intact and altering the eigenfunction as follows. If \( e^* \) is the eigenfunction associated with \( S^* \) then \( e = e^*/h \) is the Perron-Frobenius eigenfunction for \( S \). See Chen and Ludvigson (2009) and Chen et al. (2011) for semiparametric implementations of this factorization for some specific examples.

Let us turn to the specific application of Chen et al. (2011), but adopt a discrete-time counterpart to a formulation in Hansen and Scheinkman (2009). Let \( R \) be a cumulative return process and \( C \) a consumption process. Both are modeled with the “multiplicative” structure described in section 2. Form:

\[ \frac{M_t}{M_0} = \frac{R_t}{R_0} \left( \frac{C_t}{C_0} \right)^{-\gamma} \]

where \( \gamma \) is the risk aversion parameter for a power utility model of investor preferences. Chen et al. (2011) presume that the actual stochastic discount factor is:

\[ \frac{S_t}{S_0} = \exp(-\delta t) \left( \frac{C_t}{C_0} \right)^{-\gamma} \left[ \frac{h(X_t)}{h(X_0)} \right] \]

where \( X \) is the growth rate in consumption and any other variables that might forecast that growth rate. They motivate \( h \) as arising from a consumption externality in which lagged consumption is viewed as being socially determined as in Abel (1990). In contrast to Abel (1990), Chen et al. (2011) aim to be nonparametric, and \( h \) becomes a multiplicative adjustment to a marginal utility of consumption that they seek to identify with limited forward transition density to the actual transition density measured from the observed data.
restrictions. Thus they presume that the stochastic dynamics can be inferred and allow for
growth in consumption.

A fundamental result in asset pricing is that $RS$ is always a martingale, and as conse-
quence,

$$
\frac{M_t}{M_0} = \exp(\delta t) \left( \frac{\tilde{M}_t}{\tilde{M}_0} \right) \left[ \frac{h(X_0)}{h(X_t)} \right].
$$

Thus $\exp(\delta)$ is the associated Perron-Frobenius eigenvalue and $h$ is the eigenfunction. In con-
trast to Ross (2013), the actual transition dynamics are presumed to be directly identifiable.
Perhaps surprisingly $h$ and $\delta$ can both be inferred from a single return process provided that
we can infer the underlying conditional distributions from historical data.\(^{10}\) The use of mul-
tiple returns allows for the identification of $\gamma$ along with over-identifying restrictions. The
same eigenvalue and eigenfunction should be extracted when we alter the return process.\(^{11}\)

We suggest an alternative approach based on a similar idea. Let $S^*$ be a benchmark eco-
nomic model that is possibly misspecified. We no longer limit our specification $S^*$ to be the
stochastic discount factor associated with power utility. Instead we allow for a more general
starting point. We restrict the potential misspecification to have transient implications for
valuation with the actual stochastic discount factor representable as (13). The function $h$ is
introduced to capture transient sources of misspecification. We call this modification tran-
sient because the modified stochastic discount factor shares the same eigenvalue $\eta$ and hence
the same long-term interest rate, and it shares the same martingale component, which deter-
mines the long-term risk prices as argued in Hansen and Scheinkman (2009), Borovička et al.
(2011), Hansen and Scheinkman (2012b), and Hansen (2012). The presumption is that we
have little a priori structure to impose on $h$ other than limiting its long-term consequences.

Let

$$
M = RS^*
$$

and recall that $RS = \tilde{M}$ is itself a martingale. Thus we write

$$
\frac{M_t}{M_0} = \left( \frac{\tilde{M}_t}{\tilde{M}_0} \right) \left[ \frac{h(X_0)}{h(X_t)} \right].
$$

\(^{10}\)A researcher may actually prefer to place some restrictions on $h$ consistent with an underlying preference
interpretation. Even so, the identification result in Chen et al. (2011) remains interesting as additional
restrictions should only make identification easier.

\(^{11}\)In external habit models such as the one in Campbell and Cochrane (1999), the habit stock is a con-
structed state variable, one for which there is no direct observable counterpart. An additional challenge for
identification is to infer, perhaps with weak restrictions, the law of motion for this endogenous state variable.
The Perron-Frobenius eigenvalue $\exp(\eta)$ is now one and the eigenfunction $e = h$. This representation using the unit eigenvalue and the eigenfunction $h$ must hold whenever $R$ is a cumulative return process for a feasible self-financing strategy. This link across returns allows for the identification of unknown parameters needed to characterize the benchmark model $S^*$ under misspecification.

This is a rather different approach to misspecification from that suggested by Hansen and Jagannathan (1997). The approach we choose here explicitly restricts the misspecification to have transient consequences for valuation. The true $S$ and the modeled $S^*$ must share the same asymptotic decay rate (long-term interest rate) and the same martingale component. This approach does not restrict the magnitude of $h$ in any particular way. This leaves open the possibility that while $h$ has “transient” consequences for pricing, these consequences might well be very important for shorter time investment horizons. Thus in practice one would also want to assess impact of this proposed transient adjustment over investment horizons of interest in order to characterize the impact of the economic restrictions and the role for misspecification.

### 4.2 Distorted beliefs

In the previous subsection, we used a long-term perspective to introduce a structured way for economic models to fit better over longer investment horizons. One motivation for this could be a misspecification of the utility functions used to represent investor preferences, but there may be other reasons to suspect model misspecification. The stochastic discount factor specification, as we have used it so far, presumes a correct specification of the transition probabilities for $(X,Y)$ because the discount factors are only defined relative to a probability distribution. We now consider the impact of incorrectly specifying the stochastic evolution of the state variables, and we allow this to have permanent consequences on valuation. Investors themselves may have “incorrect” beliefs or they may act as if they have distorted beliefs. Motivations for this latter perspective include ambiguity aversion or investor ambitions to be robust to model misspecification. See for instance Hansen and Sargent (2001) and Chen and Epstein (2002).

To capture belief distortion as a form of model misspecification, construct

$$\frac{S_t}{S_0} = \left( \frac{S^*_t}{S^*_0} \right) \left( \frac{N_t}{N_0} \right)$$
where $S^*$ is the modeled stochastic discount factor process and $N$ is a martingale. Both processes are assumed to be of the “multiplicative” form introduced in Section 2. The process $S$ is the pertinent one for pricing assets while the process $N$ is introduced to capture distorted beliefs.

Allowing for arbitrary belief distortions is counterproductive from the perspective of building economic models. While introducing investors with preferences that display ambiguity aversion and concerns about robustness give one way to add some structure to this analysis, this approach still depends on parameters that limit the class of alternative probability models to be considered by an investor. More generally, it is of value to characterize or to limit how big is this source of misspecification from a statistical perspective. Here we consider Chernoff (1952)’s notion of entropy motivated explicitly by the difficulty in statistically discriminating between competing models. Markov counterparts to this approach rely on Perron-Frobenius theory; see Newman and Stuck (1979). The Chernoff-style calculations are “large deviation calculations” because mistakes occur when there is an unusual realization of a sequence of observations.

To define Chernoff entropy for the statistical discrimination among Markov processes, we focus on the martingale process $N$. Since $N$ is a martingale, $N^\theta$ is a supermartingale for $0 \leq \theta \leq 1$. Consider:

$$
\epsilon(\theta) = -\lim_{t \to \infty} \frac{1}{t} \log E\left[ \left( \frac{N_t}{N_0} \right)^\theta \mid X_0 = x \right].
$$

Notice that $\epsilon(0) = 0$, and since $N$ is a martingale $\epsilon(1) = 0$, but more generally $\epsilon(\theta) \geq 0$ because $N^\theta$ is a super martingale. As the asymptotic rate of growth of $\exp(\theta \log N) = N^\theta$, $\exp[-\epsilon(\theta)]$ can be seen as the Perron-Frobenius eigenvalue associated with this process. Chernoff entropy is the asymptotic decay rate for making mistakes in determining the correct model from historical data. It is given by solving:

$$
\max_{\theta \in [0,1]} \epsilon(\theta).
$$

When the maximized value is close to zero it is difficult to distinguish between the original model and the distorted model captured by $N$. Anderson et al. (2003) suggest this as a way to assess when investors concerns about model misspecification might be reasonable. More generally, this can be used to assess how large the misspecification is from a statistical standpoint.\(^{12}\) Such calculations give us a way to see if statistically small but permanent

\(^{12}\)Anderson et al. (2003) also explore a local counterpart to the Chernoff measure.
distortions in probability specifications can have notable consequence for valuation.

5 Recursive utility valuation

Recursive utility of the type initiated by Kreps and Porteus (1978) and Epstein and Zin (1989) represents the valuation of prospective future risk consumption processes through the construction of continuation values. This approach avoids the reduction of intertemporal compound lotteries and thus allows for the intertemporal composition of risk to matter. Bansal and Yaron (2004) use this feature of recursive preferences to argue that even statistically subtle components of growth rate risk can have an important impact on valuation. Hansen and Scheinkman (2012a) establish a link between Perron-Frobenius theory and Kreps and Porteus (1978) style recursive utility. This provides an alternative way to understand the valuation impacts of stochastic growth and volatility in consumption as they are compounded over time.

We use the homogeneous-of-degree-one aggregator specified in terms of current period consumption $C_t$ and the continuation value $V_t$:

$$V_t = \left[ (\zeta C_t)^{1-\rho} + \exp(-\delta) \left[ \mathcal{R}_t (V_{t+1}) \right]^{1-\rho} \right]^{1\over 1-\rho}$$  \hspace{1cm} (14)

where

$$\mathcal{R}_t (V_{t+1}) = \left( E \left[ (V_{t+1})^{1-\gamma} | \mathcal{F}_t \right] \right)^{1\over 1-\gamma}$$

adjusts the continuation value $V_{t+1}$ for risk. With these preferences, $\frac{1}{\rho}$ is the elasticity of intertemporal substitution and $\delta$ is a subjective discount rate. The parameter $\zeta$ does not alter preferences, but gives some additional flexibility that is valuable when taking limits. Next exploit the homogeneity-of-degree one specification of the aggregator (14) to obtain:

$$\left( \frac{V_t}{C_t} \right)^{1-\rho} = \zeta^{1-\rho} + \exp(-\delta) \left[ \mathcal{R}_t \left( \frac{V_{t+1} C_{t+1}}{C_{t+1} C_t} \right) \right]^{1-\rho}. \hspace{1cm} (15)$$

In finite horizons using the aggregator requires a terminal condition for the continuation value. In what follows we will consider infinite-horizon limits, leading us to consider fixed point equations. Thus we will explore the construction of the continuation value $V_t$ as a function of $C_t, C_{t+1}, C_{t+2}, \ldots.$
Suppose that the consumption dynamics evolve as:

$$\log C_{t+1} - \log C_t = \kappa(X_{t+1}, Y_{t+1}, X_t).$$

Given the Markov dynamics, we seek a solution:

$$\left( \frac{V_t}{C_t} \right)^{1-\rho} = f(X_t), \quad f \geq 0.$$ 

Writing

$$\alpha = \frac{1 - \gamma}{1 - \rho}$$

and for $f \geq 0$,

$$\mathbb{U}f = \zeta^{1-\rho} + \exp(-\delta) \left( E [f(X_{t+1})]^{\alpha} \exp\left((1 - \gamma)\kappa(X_{t+1}, Y_{t+1}, X_t) | X_t = x\right) \right)^{1\alpha},$$

we express equation (15) as:

$$\hat{f}(x) = \mathbb{U}\hat{f}(x). \quad (16)$$

The solution to the fixed point problem (16) is closely related to a Perron-Frobenius eigenvalue equation.\(^\text{13}\) To see this relation, consider the mapping:

$$Tf(x) = E \left[ \exp\left((1 - \gamma)\kappa(X_{t+1}, Y_{t+1}, X_t) | X_t = x\right) \right] f(X_{t+1}) | X_t = x. \quad (17)$$

The eigenvalue equation of interest is:

$$Te(x) = \exp(\eta)e(x) \quad (18)$$

for $e > 0$. Hansen and Scheinkman (2012a) construct a solution to (16) using the eigenfunction $e(x)$ provided the following parameter restriction

$$\delta > \frac{1 - \rho}{1 - \gamma} \eta, \quad (19)$$

is satisfied, along with some additional moment restrictions.

Why might there be a connection between the eigenvalue equation (18) and the fixed-point equation (16)? Consider equation (16), but add the following modifications:

\(^{13}\)See Duffie and Lions (1992) for a related application of Perron-Frobenius theory as an input into an existence argument.
i) Set \( \zeta^{1-\rho} = 0; \)

ii) Raise both sides of the equation to the power \( \alpha; \)

iii) Set \( \delta \) to satisfy inequality (19) with equality.

This modified version of equation (16) is essentially the same as the eigenvalue equation (18).

What do we make of this observation? The positive number \( \zeta \) is merely a convenient scale factor. Its specific choice does not alter the preferences implied by the recursive utility model. This gives us some flexibility in taking limits. The limit that we must take entails letting the subjective discount rate \( \delta \) approximate the bound in (19). In effect, this makes the future as important as possible in the utility recursion. Hansen and Scheinkman (2012a) discuss more formally how this limit effectively reduces the infinite horizon value function problem to a Perron-Frobenius eigenvalue problem. For many common model parameterizations, the eigenfunction is the exponential of a quadratic and the eigenvalue equation can be solved in a straightforward manner. This solution then can be used to construct a solution to the infinite horizon utility recursive under the stated parameter restrictions.

In the notation of the previous Sections, the process \( M_t \) used in conjunction with the Perron-Frobenius theory is

\[
\frac{M_t}{M_0} = \left( \frac{C_t}{C_0} \right)^{1-\gamma}.
\]

The martingale component for this process provides a convenient change of measure to use for evaluating the utility recursion as it absorbs the stochastic growth component to consumption pertinent for valuation. This approach thus features the risk aversion parameter \( \gamma \) in the construction of a martingale component relevant for the analysis of investor preferences.

6 Conclusion

In this paper we studied three problems using a common approach. This approach assumes a Markov environment and applies Perron-Frobenius theory to characterize the long-run impact of compounding in a stochastic environment. Given a possibly non-linear stochastic process describing growth or discounting, this approach identifies three multiplicative components: i) a deterministic growth or decay rate rate; ii) a transient component and iii) a martingale component. The latter two components are typically correlated. We use the martingale component to produce a distorted probability measure that helps reveal long-term contribution to valuation induced by stochastic growth or discounting.
First, we applied these methods to investigate how compounding interacts with the assignment of risk prices and to examine the possibility of disentangling risk adjustments from investor beliefs using asset market data. We showed that, in the presence of stochastic growth, application of Perron-Frobenius theory does not recover the actual probability used by investors but instead recovers the distorted probability measure. This measure is informative about the implications for risk prices over long payoff and investment horizon.

Second, we used the decomposition to examine misspecification of parametric models of valuation. These models, while tractable, are surely misspecified. Departures from economic fundamentals may be of a temporary nature, and econometric identification should reflect this possibility. This can be accomplished in the framework developed in this paper by assuming that the true stochastic discount factor and the parametric model of the discount factor share the same deterministic growth rate and martingale component, differing only in their respective transient components to valuation. We also discussed how to treat the impact of assuming that investors have “wrong” beliefs in environments in which we allow for the misspecification to have permanent consequences for valuation.

Third, we discussed how to use Perron-Frobenius theory to obtain existence results for infinite-horizon Kreps-Porteus utility functions, and highlighted the role of the risk-aversion parameter in the construction of the stochastic growth component of consumption that is pertinent to long-run valuation.

The methods we describe here can also be applied to study the valuation of unusual episodes as they emerge over multiple time periods. While the episodes might be disguised in the short run, they could become more prominent over longer horizons. In our analysis here, we applied a generalized version of Perron-Frobenius theory for Markov processes, an approach that has much in common with large deviation theory as developed by Donsker and Varadhan (1976). To better appreciate this connection, see Stutzer (2003) and Hansen and Scheinkman (2012a). An alternative approach analyzes such phenomena with a different type of limiting behavior. In future work, we plan to study the pricing rare events using a large-deviation theory in continuous time whereby we hold fixed the valuation time interval while progressively reducing the exposure to Brownian motion shocks.
A Mixture of Normals example

Consider next an alternative approach to recovery for Example 3.2. The positive random variable:

\[
\frac{S_{t+1}}{S_t} \cdot E\left( \frac{S_{t+1}}{S_t} | X_{t+1}, X_t \right)
\]

alters the distribution of \(Y_{t+1}\) conditioned on \(X_{t+1}\) and \(X_t\) in the same manner as described previously. The adjustment induced by:

\[
E\left( \frac{S_{t+1}}{S_t} | X_{t+1}, X_t \right)
\]

is captured in the \(A\) matrix. Consider now a payoff of the form \(g_1(Y_{t+1})g_2(X_{t+1}, X_t)\) where \(g_2\) is one when \(X_t\) is in state \(i\) and \(X_{t+1}\) is in state \(j\). We may compute this price by first evaluating the distorted expectation of \(g_1\) conditioned on \(X_{t+1}, X_t\) and then multiplying this conditional expectation by \(a_{ij}\). Since the \(a_{ij}\)'s are can be inferred by asset market data so can the altered distribution for \(Y_{t+1}\) conditioned on \(X_{t+1}, X_t\). Thus \(\tilde{\alpha}\) is revealed. The terms \(\tilde{\beta} \cdot \mathbf{u}_i\) and the \(p_{ij}\)'s can be solved in terms of the \(a_{ij}\)'s and the \(\frac{1}{2}(u_j)'\tilde{\alpha}'\tilde{\alpha}u_j\)'s. Thus \(P\) can be recovered but not by a Perron-Frobenius extraction. In the more general case:

\[
\log S_{t+1} - \log S_t = (X_t)'\tilde{\beta}X_{t+1} + (X_{t+1})'\tilde{\alpha}'Y_{t+1}
\]

where \(\tilde{\beta}\) is an \(n \times n\) matrix then it is typically not possible to recover \(P\) without additional restrictions.

B Markov diffusion example

In this section we give some supporting arguments for the discussion of Example 3.3. Suppose that \(X\) is a Markov diffusion:

\[
dX_t = \mu(X_t)dt + \Lambda(X_t)dW_t,
\]

and

\[
d\log S_t = \beta(X_t)dt + \alpha(X_t) \cdot dW_t.
\]

To construct a transformed generator we compute the drift of \(\exp(-\eta t) f(x_t) \left( \frac{e(x_t)}{e(X_0)} \right)\).
This gives rise to the transformed generator:

\[
\tilde{G}(f)(x) = \frac{1}{e(x)}[G(ef)](x) - \eta f(x)
\]

\[
= [Gf](x) + \frac{f(x)}{e(x)}[G(e)](x) - \left[\beta(x) + \frac{1}{2}|\alpha(x)|^2\right] f(x)
\]

\[
+ \left[\frac{\partial \log e(x)}{\partial x}\right]' \Lambda(x) \Lambda(x)' \left[\frac{\partial f(x)}{\partial x}\right] - \eta f(x)
\]

\[
= [Gf](x) - \left[\beta(x) + \frac{1}{2}|\alpha(x)|^2\right] f(x) + \left[\frac{\partial \log e(x)}{\partial x}\right]' \Lambda(x) \Lambda(x)' \left[\frac{\partial f(x)}{\partial x}\right]
\]

\[
= \left[\mu(x)' + \alpha(x)' \Lambda(x)' \left[\frac{\partial \log e(x)}{\partial x}\right]' \Lambda(x) \Lambda(x)' \left[\frac{\partial f(x)}{\partial x}\right]
\]

\[
+ \frac{1}{2} \text{trace} \left[\Lambda(x) \Lambda(x)' \frac{\partial^2 e(x)}{\partial x \partial x'}\right].
\]

As expected, the coefficient on the level \( f(x) \) is zero, because we want this to be the generator of a stochastically stable Markov process. The matrix of coefficients for the second derivative of \( f \) remains the same as the absolute continuity between the implied probability measures over finite time intervals requires that the diffusion matrix remain the same. The important change is in the vector of coefficients for the first derivative of \( f \). This modification alters the drift of the diffusion in the manner state in the discussion of Example 3.3.

This approach recovers the original generator of the Markov process when \( e \) solves the eigenfunction equation (12) and:

\[
\alpha(x) = -\Lambda(x)' \left[\frac{\partial \log e(x)}{\partial x}\right].
\]

Given these restrictions, it follows that

\[
\eta = \beta(x) - \frac{1}{2} \left[\frac{\partial \log e(x)}{\partial x}\right]' \Lambda(x) \Lambda(x)' \left[\frac{\partial \log e(x)}{\partial x}\right]
\]

\[
+ \mu(x)' \frac{\partial \log e(x)}{\partial x} + \frac{1}{2e(x)} \text{trace} \left[\Lambda(x) \Lambda(x)' \frac{\partial^2 e(x)}{\partial x \partial x'}\right]
\]

\[
= \beta(x) + \mu(x)' \frac{\partial \log e(x)}{\partial x} + \frac{1}{2} \text{trace} \left[\Lambda(x) \Lambda(x)' \frac{\partial^2 \log e(x)}{\partial x \partial x'}\right].
\]
As a consequence, $\beta(x) = \eta - \text{drift } \log e(x)$ where

$$\text{drift } \log e(x) = \mu(x)' \frac{\partial \log e(x)}{\partial x} + \frac{1}{2} \text{trace} \left[ \Lambda(x) \Lambda(x)' \frac{\partial^2 \log e(x)}{\partial x \partial x'} \right]$$

and hence

$$d \log S_t = \eta dt - d \log e(X_t) dt.$$ Integrating between dates zero and $t$ and exponentiating gives:

$$\frac{S_t}{S_0} = \exp(\eta t) \left[ \frac{e(X_0)}{e(X_0)} \right].$$
References


