Additive Representation for Preferences over Menus in Finite Choice Settings*

Leandro Gorno†
Princeton University

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Abstract

This paper proves that the additive representation of Dekel-Lipman-Rustichini (2001) is consistent with any preference relation among the deterministic alternatives in their model. The result yields an additive representation which relaxes both the monotonicity and ordinal submodularity axioms in Kreps (1979) flexibility representation theorem.

KEYWORDS: Dynamic choice; Preference for flexibility; Preference for commitment

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† Department of Economics, Fisher Hall, Princeton University, Princeton NJ 08544 USA, (email: lgorno@princeton.edu)
1. Introduction

The pioneering work of Kreps [13] introduced preferences over menus and obtained an additive representation theorem that rationalizes preference for flexibility as subjective uncertainty over future tastes. His contribution motivated the papers by Gul and Pesendorfer [7] (GP) and Dekel, Lipman and Rustichini [1] (DLR), which subsequently triggered a prolific literature on dynamic choice.

While Kreps studies a finite choice setting in which a decision maker (DM) first chooses a menu with deterministic alternatives and then selects one of the alternatives contained in that menu, DLR (and GP) obtain their representation by introducing lotteries (i.e. probability distributions over the alternatives) and considering preferences over menus of those lotteries. However, as observed by Olszewski [21], most of the examples (if not all) in this literature refer to finite choice situations in which lotteries seem to play no essential role. Therefore, it is natural to ask whether the DLR axioms imply any constraints on finite choice behavior and, if so, what are exactly these constraints. This paper provides a negative answer by showing that every preference over menus of finitely many alternatives is consistent with the DLR representation. In Kreps’ original setting, the result implies a generalization of his flexibility representation theorem which relaxes both the monotonicity and the ordinal sub-modularity axioms.

The paper is structured as follows: Section 2 presents the main result, Section 3 relates it to the existing literature and Section 4 offers a conclusion. To simplify the exposition, all proofs are collected in the Appendix.

[1] The topics studied include: preference for flexibility [1], [3], [13], [15], [23], temptation and self-control [1], [2], [3], [7], [8], [11], [17], [18], [19], [20], [21], guilt [4], perfectionism [10], self-deception [12], regret [24], contemplation costs [5], [6] and thinking aversion [22].
2. Model and main result

Let $X$ be a non-empty finite set, define $\mathcal{A}$ to be the set of all non-empty subsets of $X$ and let $\succeq$ be a generic binary relation on $\mathcal{A}$ (with $>$ and $\sim$ standing for its asymmetric and symmetric parts, respectively). Kreps (13) uses this simple setting to represent a DM that faces a two-stage decision process. In the first stage, she chooses a menu $A \in \mathcal{A}$. In the second stage, she chooses an alternative $x \in A$ from the previously chosen menu.

The alternatives in Kreps’ model are deterministic\(^2\). DLR (1), in turn, use a richer setting in which the DM chooses menus of lotteries. Formally, let $\Delta(X)$ be the set of probability measures on $X$ (lotteries) endowed with the Euclidean topology. Let $\mathcal{A}^*$ be the set of closed (hence compact) subsets of $\Delta(X)$ and let $\succeq^*$ be a generic binary relation on $\mathcal{A}^*$ (with $>^*$ and $\sim^*$ standing for its asymmetric and symmetric parts, respectively).

For any deterministic alternative $x \in X$, denote by $\delta_x \in \Delta(X)$ the degenerate lottery that assigns probability 1 to $x$. For any menu of deterministic alternatives $A \in \mathcal{A}$, define the lottery menu $\delta(A) := \{\delta_x | x \in A\}$. The binary relation $\succeq^*$ over $\mathcal{A}^*$ is said to be an extension of the binary relation $\succeq$ over $\mathcal{A}$ if $A \sim B$ implies $\delta(A) \sim^* \delta(B)$ and $A > B$ implies $\delta(A) >^* \delta(B)$.

The binary relation $\succeq$ (resp. $\succeq^*$) is said to be represented by $U \in \mathbb{R}^\mathcal{A}$ (resp. $U^* \in \mathbb{R}^{\mathcal{A}^*}$) if $x \succeq y \Leftrightarrow U(x) \geq U(y)$ (resp. $p \succeq^* q \Leftrightarrow U^*(p) \geq U^*(q)$). $\succeq$ is called a preference if it is complete and transitive.

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\(^2\) Of course, nothing changes if $X$ is a finite set of lotteries.
The binary relation $\succeq^*$ is said to be a DLR preference if there are finite sets $S_1, S_2$ and a function $u \in \mathbb{R}^{X \times (S_1 \cup S_2)}$ such that the function $U^* \in \mathbb{R}^{\mathcal{A}^*}$ defined by

$$U^*(A^*) := \sum_{s \in S_1} \max_{p \in \mathcal{A}^*} \left( \sum_{x \in X} u(x, s)p(x) \right) - \sum_{s \in S_2} \max_{p \in \mathcal{A}^*} \left( \sum_{x \in X} u(x, s)p(x) \right) \quad A^* \in \mathcal{A}^*$$

represents $\succeq^*$. Every DLR preference over $\mathcal{A}^*$ induces a preference over $\mathcal{A}$ by associating each deterministic alternative with the corresponding degenerate lottery. That the converse is also true is the main result of this paper:

**Theorem 1.** Every preference on $\mathcal{A}$ can be extended to a DLR preference on $\mathcal{A}^*$.

Whenever $\succeq^*$ is an extension of a preference $\succeq$ on $\mathcal{A}$, both $\succeq^*$ and $\succeq$ imply the same choice behavior among menus of deterministic alternatives (for it follows that $A \succeq B$ if and only if $\delta_A \succeq^* \delta_B$ for all $A, B \in \mathcal{A}$).

Note that Theorem 1 makes no claim of uniqueness. This is to be expected since, as observed by DLR [1] among others, it is not possible to identify the states in Kreps’ representation result. In the context of Theorem 1, this lack of identification entails the generic existence of multiple DLR extensions of the same deterministic preference. It should be stressed that, while lack of uniqueness might decrease the appeal of using the finite setting for modeling behavior, the purpose of this paper is not to argue that we should do so, but rather to show the (lack of) finite choice implications of assuming the DLR axioms in the lottery setting.
3. Relation to the literature

Kreps [13] introduced the preference-over-menus approach focusing on preference for flexibility in a finite choice setting. Specifically, he showed that a preference relation $\succeq$ on $\mathcal{A}$ admits a representation of the form

$$U(A) = \sum_{s \in S} \max_{x \in A} u(x, s)$$

if and only if $\succeq$ satisfies two axioms, namely set monotonicity (SM) and ordinal submodularity (OSM), defined as:

(SM) $A \supseteq B$ implies $A \succeq B$

(OSM) $A \sim A \cup B$ implies $A \cup C \sim A \cup B \cup C$

It is immediate that Theorem 1 implies the following:

**Corollary 1.** Every preference on $\mathcal{A}$ can be represented by a function $U \in \mathbb{R}^{\mathcal{A}}$ which can be written:

$$U(A) = \sum_{s \in S_1} \max_{x \in A} u(x, s) - \sum_{s \in S_2} \max_{x \in A} u(x, s)$$

for some finite sets $S_1$ and $S_2$, and a function $u \in \mathbb{R}^{X \times (S_1 \cup S_2)}$.

Corollary 1 extends Kreps’ representation by allowing for “negative” states and relaxing both SM and OSM. It follows that, while Kreps’ axioms effectively identify those preferences which can be represented in an additive fashion without resorting to negative states, no further weakening is needed to obtain just the additivity.
DLR [1] and Dekel, Lipman, Rustichini and Sarver [3] proved a generalization of Kreps’ result in the lottery setting. Specifically, they characterized a representation of the form:

\[ U^*(A^*) = \int_S \sup_{x^* \in A^*} u^*(x^*, s) \mu(ds) \]

where \( u^* \in \mathbb{R}^{A(x) \times S} \) has the expected utility form, \( S \) might be infinite and \( \mu \) is a signed measure. Consistently with Kreps’ result, DLR also prove that, under SM, \( \mu \) is positive. Obviously, restricting \( U^* \) to degenerate lotteries yields the following representation on \( \mathcal{A} \):

\[ U(A) = \int_S \max_{x \in A} u(x, s) \mu(ds) \]

DLR later refined their representation result to make the state space \( S \) finite (see [2], Theorem 6), effectively characterizing the class of DLR preferences as defined in the previous section\(^3\). DLR preferences were further studied in the lottery setting by Kopylov [9], who obtained conditions determining the number of positive and negative components.

The results presented in this paper also shed some light on the individual role of substantive axioms in the literature. For instance, preferences satisfying SM but not OSM have been studied by Ergin [5] and Natzenzon [15] in the finite setting. Theorem 1 shows that every such a preference has an extension to the lottery setting. However, this extension cannot satisfy SM for OSM would follow\(^4\). It follows that Kreps’ OSM assumption characterizes those preferences which can be extended to the lottery setting preserving the desire for flexibility.

\(^3\) Due to finiteness of the state space, DLR called the representation involved a “finitely additive representation”.

\(^4\) DLR prove that SM and independence imply OSM (see (1), footnote 21).
Moreover, the representation in Corollary 1 also relates to GP's temptation and self-control representation in [7]. Specifically, GP obtain a representation on $\mathcal{A}^*$ of the form:

$$U^*(A^*) := \max_{x^* \in \mathcal{A}^*} \left( w^*(x^*) - \max_{y^* \in \mathcal{A}^*} \left( v^*(y^*) - v^*(x^*) \right) \right)$$

where the functions $w^*$ and $v^*$ have expected utility form, $w^*$ is interpreted as a normative utility ranking and the inner maximized term as the cost of self-control. It is easy to verify that this is a particular case of DLR's representation with one positive state and possibly one negative state. GP (8) later explored finite analogues, but the representations they obtained are non-additive and rely on axioms which are less transparent than those employed by the same authors in the lottery setting.

Finally, in a recent contribution, Stovall [25] provided axioms relaxing those of GP [7] such that a preference on $\mathcal{A}^*$ can be represented by:

$$U^*(A^*) := \sum_{s \in S} \max_{x^* \in \mathcal{A}^*} \left( w^*(x^*) - \max_{y^* \in \mathcal{A}^*} \left( v^*(y^*, s) - v^*(x^*, s) \right) \right) \pi(s)$$

The interpretation proposed by Stovall is that of uncertain temptations. Similarly to the case of DLR, one may wonder how this representation constraints finite choices. A partial answer is given by the following:

**Corollary 2.** For every function $U \in \mathbb{R}^\mathcal{A}$ there is a finite set $S$, a positive measure $\pi$ on $S$ and functions $v, w \in \mathbb{R}^{\mathcal{A} \times S}$ such that

$$U(A) = \sum_{s \in S} \max_{x \in \mathcal{A}} \left( w(x, s) - \max_{y \in \mathcal{A}} \left( v(y, s) - v(x, s) \right) \right) \pi(s)$$
Corollary 2 is a generalized finite analogue of Stovall’s result in which the normative utility is also random. The lack of any constraint on preferences means that all the substantive restrictions on finite choice behavior in Stovall’s representation are embodied in the state-independence of the normative utility.

4. Concluding remarks

The literature on preferences over menus typically models DMs who care about lotteries even though, most often than not, only their deterministic choices are of real interest. In these cases, there is a gap between what the axioms talk about and the relevant content of the theories. To bridge that gap, Theorem 1 characterizes the finite deterministic choice behavior associated with DLR’s additive representation by showing that every preference relation over the set of all menus of a finite set can be extended to the lottery setting ensuring that all DLR axioms are satisfied. It follows that DLR’s DMs are not restricted in this respect beyond the standard requirements of completeness and transitivity which are necessary for any representation.

As a final comment, I want to stress that the main point of this analysis is to shed light on how exactly DLR and related models constrain finite choice behavior, not to argue that they are too weak to be useful or that the lottery setting should be abandoned. Theorem 1 constitutes a formal verification that DLR do not surreptitiously forbid choice behavior that would be allowed if lotteries were not available to empower their axioms. In this sense, it allows one to conclude that the linear lottery structure used by DLR to identify the state space does not sacrifice generality regarding finite deterministic choices.
Appendix: Proofs

The following lemma is used in the proof of Theorem 1 below:

**Lemma 1.** Every function \( U \in \mathbb{R}^\mathcal{A} \) can be written

\[
U(A) = \sum_{s \in S_1} \max_{x \in A} u(x, s) - \sum_{s \in S_2} \max_{x \in A} u(x, s)
\]

for some finite sets \( S_1 \) and \( S_2 \), and a function \( u \in \mathbb{R}^{\times (S_1 \cup S_2)} \).

**Proof.** Fix an arbitrary function \( U \in \mathbb{R}^\mathcal{A} \). Define \( \phi(\emptyset) := 0 \) and \( \phi(A) := 1 \) for any \( A \in \mathcal{A} \). The conjugate Möbius transform is a bijection on \( \mathbb{R}^\mathcal{A} \) (see Lemma 1 in Nehring [16]). More specifically, one can write \( U \) as

\[
U(A) = \sum_{s \in \mathcal{A}} \lambda(s) \phi(s \cap A)
\]

where

\[
\lambda(s) := \sum_{B \subseteq s} (-1)^{\#(s \setminus B) + 1} U(X \setminus s)
\]

Define \( S_1 := \{ s \in \mathcal{A} | \lambda(s) > 0 \} \), \( S_2 := \{ s \in \mathcal{A} | \lambda(s) < 0 \} \) and \( u \in \mathbb{R}^{\times (S_1 \cup S_2)} \) by

\[
u(x, s) := \begin{cases} \lvert \lambda(s) \rvert & x \in s \\ 0 & x \notin s \end{cases}
\]

Then, one can verify that, for every \( A \in \mathcal{A} \),

\[
U(A) = \sum_{s \in S_1} \max_{x \in A} u(x, s) - \sum_{s \in S_2} \max_{x \in A} u(x, s)
\]

proving the claim \( \blacksquare \)
Theorem 1. Every preference on $\mathcal{A}$ can be extended to a DLR preference on $\mathcal{A}^*$. 

Proof. Let $\succeq$ be a preference relation on $\mathcal{A}$. It is well known that every preference relation is representable. Let $U \in \mathbb{R}^d$ be a representation. By Lemma 1, $U$ can be written in the form:

$$U(A) = \sum_{s \in S_1} \max_{x \in A} u(x, s) - \sum_{s \in S_2} \max_{x \in A} u(x, s) \quad \forall A \in \mathcal{A}$$

Then, extend $u \in \mathbb{R}^{X \times (S_1 \cup S_2)}$ to $u^* \in \mathbb{R}^{\Delta(X) \times (S_1 \cup S_2)}$ by defining:

$$u^*(x^*, s) := \sum_{x \in X} u(x, s)x^*(x) \quad \forall x^* \in \Delta(X), \ s \in S_1 \cup S_2$$

Hence, one can define $U^* \in \mathbb{R}^{d^*}$ by setting

$$U^*(A^*) := \sum_{s \in S_1 \setminus A^*} \max_{x^* \in A^*} u^*(x^*, s) - \sum_{s \in S_2} \max_{x^* \in A^*} u^*(x^*, s) \quad \forall A^* \in \mathcal{A}^*$$

Note that $U^*$ is well-defined, since each function $u^*(\cdot, s)$ is a linear function in a finite-dimensional space, hence continuous. Finally, define $\preceq^*$ on $\mathcal{A}^*$ by $A^* \preceq^* B^* \iff U^*(A^*) \geq U^*(B^*)$. By definition, $\preceq^*$ is a DLR preference. Moreover, for every $A \in \mathcal{A}$,

$$U^*(\delta^*_A) = \sum_{s \in S_1 \setminus \delta^*_A} \max_{x^* \in \delta^*_A} u^*(x^*, s) - \sum_{s \in S_2} \max_{x^* \in \delta^*_A} u^*(x^*, s)$$

$$= \sum_{s \in S_1 \setminus \delta^*_A} \max_{x^* \in \delta^*_A} u^*(\delta^*_x, s) - \sum_{s \in S_2} \max_{x^* \in \delta^*_A} u^*(\delta^*_x, s)$$

$$= \sum_{s \in S_1 \setminus \delta^*_A} \max_{x \in A} u(x, s) - \sum_{s \in S_2} \max_{x \in A} u(x, s) = U(A)$$

Therefore, $\succeq^*$ extends $\succeq$. ■
**Corollary 1.** Every preference on \( \mathcal{A} \) can be represented by a function \( U \in \mathbb{R}^\mathcal{A} \) which can be written:

\[
U(A) = \sum_{s \in S_1} \max_{x \in A} u(x, s) - \sum_{s \in S_2} \max_{x \in A} u(x, s)
\]

for some finite sets \( S_1 \) and \( S_2 \), and a function \( u \in \mathbb{R}^{\times(S_1 \cup S_2)} \).

**Proof.** This is just the first step in the proof of Theorem 1. \( \blacksquare \)

**Corollary 2.** For every function \( U \in \mathbb{R}^{\mathcal{A}} \) there is a finite set \( S \), a positive measure \( \pi \) on \( S \) and functions \( v, w \in \mathbb{R}^{\times S} \) such that

\[
U(A) = \sum_{s \in S} \max_{x \in A} \left( w(x, s) - \max_{y \in A} (v(y, s) - v(x, s)) \right) \pi(s)
\]

**Proof.** By Lemma 1, it is possible to write

\[
U(A) = \sum_{s \in S_1} \max_{x \in A} u(x, s) - \sum_{s \in S_2} \max_{x \in A} u(x, s)
\]

where \( S_1 \) and \( S_2 \) are finite and \( u \in \mathbb{R}^{\times(S_1 \cup S_2)} \).

Define the finite set \( S := S_1 \cup S_2 \), a “uniform” probability measure \( \pi \in \Delta(S) \) by setting \( \pi(s) := 1/|S| \) and functions \( v, \hat{w}, w \in \mathbb{R}^{\times S} \) by

\[
v(x, s) := \begin{cases} 0 & s \in S_1 \\ \frac{u(x, s)}{|S|} & s \in S_2 \setminus S_1 \end{cases} \quad \hat{w}(x, s) := \begin{cases} \max_{y \in A} (v(y, s) - v(x, s)) & s \in S_1 \\ 0 & s \in S_2 \end{cases} \quad w(x, s) := \hat{w}(x, s) - v(x, s) \quad s \in S
\]
Then, $U$ can be written

$$U(A) = \sum_{s \in S_1} \max_{x \in A} \hat{w}(x, s) \pi(s) - \sum_{s \in S_2} \max_{x \in A} \nu(x, s) \pi(s)$$

$$= \sum_{s \in S} \left( \max_{x \in A} \hat{w}(x, s) - \max_{x \in A} \nu(x, s) \right) \pi(s)$$

$$= \sum_{s \in S} \left( \max_{x \in A} (w(x, s) + \nu(x, s)) - \max_{y \in A} \nu(y, s) \right) \pi(s)$$

$$= \sum_{s \in S} \max_{x \in A} \left( w(x, s) - \max_{y \in A} (\nu(y, s) - \nu(x, s)) \right) \pi(s)$$

as claimed \[\blacksquare\]
References


