Optimality of Securitized Debt with Endogenous and Flexible Information Acquisition

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An issuer designs a contract and agents flexibly acquire information when deciding whether to accept it and provide liquidity. Unlike the existing literature, we do not impose any physical restriction on information structure to capture the idea of flexible information acquisition. Facing an informational cost measured by reduction of Shannon’s entropy, agents collect the most relevant information that is determined by the "shape" of the contract. Thus in order to reduce the effect of adverse selection, the issuer designs the contract to avoid information acquisition, or induce her counterparty to acquire information least harmful to her interest. Without restricting attention to monotone securities, we show that pooling and tranching (i.e., issuing securitized debt) is uniquely optimal in providing liquidity, regardless of the stochastic interdependence of underlying assets and the allocation of bargaining power.

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Pooling assets and issuing a senior tranche is a popular way to raise liquidity. For example, commercial banks pool a large number of individual home mortgages or automobile loans to create a Special Purpose Vehicle (SPV), which then issues asset-backed securities (ABSs) to finance the purchase of these loans. According to the security design literature, this securitization process helps reduce the effect of adverse selection and thus enhances liquidity provision. Most models in the literature capture this idea by assuming that the seller owns private information about the assets that makes potential buyers hesitant to purchase, such as (David C. Nachman and Thomas H. Noe 1994), (Peter M. DeMarzo and Darrell Duffie 1999) and (Peter M. DeMarzo 2005). All these models have exogenous information structures. They treat the agents as insiders observing private signals that covary with fundamentals in an exogenously specified way. That is, agents make decisions according to their information but cannot decide what information to acquire.

This paper explores another source of information asymmetry, i.e., information acquisition. We follow (Tri Vi Dang, Gary Gorton and Bengt Holmstrom 2011) in treating the agents as experts who acquire information. Empirically, the agents (traders, credit rating agencies, etc.) involved in ABS transactions are skillful and sophisticated. Their expertise in assessing investment opportunities is better modeled by endogenous rather...
than exogenous information (acquisition). Here endogeneity means that the agents can choose from a set of information structures according to their investment opportunities. Taking this endogeneity into account, the issuer designs securities least sensitive to the buyer’s information acquisition. (Dang, Gorton and Holmstrom 2011) show that debt is the least information-sensitive and thus is an optimal contract to provide liquidity. However, there also exist infinitely many other securities as information-sensitive as the debt contract. They also identify conditions under which securitization is optimal.

This paper differs from (Dang, Gorton and Holmstrom 2011) in flexible information acquisition. Here flexibility means that the agents’ choice set of information structures consists of all conditional distributions of signals given the state of the world. It captures the ability of agents to allocate their attention in whatever way they want. We model flexible information acquisition through the framework of rational inattention (e.g. (Christopher A. Sims 2003)), where choosing an information structure incurs a cost measured by the amount of information conveyed by that information structure. This cost represents the time or resource that is consumed to run models, do statistical tests or write reports. Since a piece of information is useful only if it helps evaluate some investment opportunity, our agents only collect the most relevant information. For example, to assess a collateralized debt with face value $1000 and price $800, an potential buyer would collect data and test if the underlying asset is worth $800 or not. She should not waste her time and resource to just figure out if the collateral would reach $1500 or $2000. Similar to (Dang, Gorton and Holmstrom 2011), debt is optimal in our model. But our result is sharper in the sense that the debt contract is the uniquely optimal one. Moreover, flexible information acquisition provides a unified framework to analyze securitization. We show that issuing securitized debt (i.e. pooling and tranching) is uniquely optimal to raise liquidity, regardless of the stochastic interdependence among underlying assets and the allocation of bargaining power.

Another reason to introduce endogenous and flexible information acquisition is the concern of robustness. Most papers in the existing literature derive their results under specific assumptions about information structure. For example, (DeMarzo and Duffie 1999) assumes the existence of a uniform worst case, (Peter M. DeMarzo, Ilan Kremer and Andrzej Skrzypacz 2005) requires signals to satisfy the Monotone Likelihood Ratio Property (MLRP), and (DeMarzo 2005) imposes an additive separable relation between cash flows and signals. Our framework avoids the arbitrariness in assuming information structure since we do not impose any physical restriction on the choice set of information structures. Moreover, we show that MLRP naturally emerges in equilibrium.

In our leading example, a risk-neutral seller owns some assets generating uncertain future cash flows. She is impatient and wants to raise liquidity by issuing asset-backed securities to a risk-neutral buyer. To focus on the adverse selection resulting from endogenous information acquisition, we assume there is no ex ante information asymmetry. Thus the underlying cash flows are distributed according to a common prior. To raise liquidity, the seller proposes an ABS and its price, and sets it as a take-it-or-leave-it offer. Facing this offer, the buyer acquires information about the underlying assets and then makes a decision. Information acquisition is costly, but flexible. There is no physical re-
striction on the information structure, but the one that conveys more information is more expensive. The buyer’s incentive to acquire information is shaped by the offer facing her. Thus the seller designs the ABS to avoid information acquisition or induce information acquisition least harmful to her own interests. Due to the limited liability, the ABS to be designed is bounded above by the sum of underlying cash flows. In the case that information acquisition is not expensive, the buyer attempts to distinguish any states with different payoffs. Hence the seller makes the ABS constant whenever it is off the boundary to avoid information acquisition and thus adverse selection. When the cash flows are too low to support such constant, the ABS reaches the boundary and equals the sum of underlying cash flows. Therefore, a securitized debt becomes the uniquely optimal ABS in presence of endogenous and flexible information acquisition. When information acquisition is expensive, the seller may find it optimal to give the buyer no incentive to acquire any information. This requires the seller to design the repayment of the security as "constant" as possible. A securitized debt again turns out to be uniquely optimal in this case.

We model players’ information acquisition behavior through the framework of rational inattention. The basic idea of rational inattention is that people face informational capacity constraints defined by Shannon’s information theory. That is, there are limited bits that can be used to reduce the subjective uncertainty of some exogenous variables. As a result, players have to pay attention to those aspects most relevant to their welfare and rationally ignore other information. (Christopher A. Sims 1998) pioneers to introduce rational inattention to model price stickiness, where the capacity constraints dampen and delay people’s responses to shocks. (Sims 2003) and (Christopher A. Sims 2005) further develop the theory to accommodate dynamic programming in both linear-quadratic and non-linear-quadratic cases. Later (Filip Matejka 2010) shows that a perfectly attentive seller sets discrete and rigid prices to stimulate a rationally inattentive buyer to consume more. Recently, (Bartosz Mackowiak and Mirko Wiederholt 2009) examine Rational Inattention in a dynamic stochastic general equilibrium model. They show that prices respond strongly and quickly to idiosyncratic shocks but weakly and slowly to aggregate shocks.

In applied work, rational inattention is mainly studied in two cases: the linear-quadratic case (e.g., (Mackowiak and Wiederholt 2009)), and the binary-action case. A leading example of the latter is (Michael Woodford 2009), where firms acquire information and then decide whether to reduce their prices. Rather than focusing on decision problems, (Ming Yang 2011) studies information acquisition in strategic situations. (Yang 2011) shows that introducing endogenous and flexible information acquisition dramatically changes predictions of binary-action coordination games with incomplete information. Our model can also be viewed as binary-action game with strategic information acquisition. But in our model, players move sequentially rather than simultaneously as in (Yang 2011).

The security design literature has contributed a lot to our understanding of asset-backed securities, but our findings vary due to different approaches. (Gary Gorton and George G. Pennacchi 1990) shows that splitting assets into debt and equity mitigates the
lack problem between outsiders and insiders. They directly assume the existence of debt rather than considering a security design problem. In (DeMarzo and Duffie 1999), informed sellers signal the quality of assets to competitive liquidity suppliers through retaining part of the cash flows. Equity is issued when the contractible information is not very sensitive to sellers’ private information. Standard debt is optimal within the set of non-decreasing securities if the information structure allows a uniform worst case. (Bruno Biais and Thomas Mariotti 2005) studies the effects of market power on market liquidity. They derive both the optimal security and trading mechanism through the approach of mechanism design. Debt contract turns out to be optimal under distributional conditions of underlying cash flows. (DeMarzo 2005) focuses on the consequences of pooling and tranching. Pooling has an information destruction effect that destroys the seller’s ability to signal the quality of her assets separately. When tranching is possible, pooling may also have a risk diversification effect that reduces informational sensitivity of the senior claim. Under specific distributional assumptions of the noise structure, (DeMarzo 2005) shows that the risk diversification effect dominates the information destruction effect as the number of underlying assets goes to infinity. In this limit case, pooling and tranching become optimal.

We classify the above models as security design with exogenous information structure. Their results are limited for two reasons. First, many restrictions are imposed on the information structure. For example, (DeMarzo and Duffie 1999) and (DeMarzo 2005) assume the existence of a uniform worst case. (DeMarzo 2005) also requires additive noise with log-concave density. In (Biais and Mariotti 2005), the signal is assumed to be the cash flow itself. Second, all these models restrict their attention to non-decreasing securities. (Robert D. Innes 1990) provides a standard motivation for this constraint. When the security is not monotone, a seller may cheat through borrowing from a third party, reporting a high cash flow to reduce her repayment and then repaying the side loan. The validity of this argument depends on the context. In the case of publicly traded stocks or bonds, this kind of cheat is unlikely to happen because it is difficult or even illegal for seller to manipulate the cash flows. Moreover, when the security is written on multiple underlying assets, even the concept of monotonicity is not well defined. Our framework is free of these limits. We have no restriction on the information structure since it is endogenously determined. We do not impose any constraint on the securities except the standard feasibility condition. Through this model, pooling all assets and then issuing a debt backed by this pool is always optimal regardless of the stochastic interdependence among underlying assets. This is not the same as the limit optimal result in (DeMarzo 2005). It holds for any number of underlying assets.

The most related work is characterized by endogenous but rigid information acquisition. (Dang, Gorton and Holmstrom 2011) establish an informational sensitivity criterion to link the current theories of debt to the origins of financial crises. They show that debt
is least information-sensitive in presence of information acquisition. During normal period, debt contracts generate least incentive for agents to acquire private information and thus enhance liquidity provision. However, debt is also vulnerable to bad news which may trigger information acquisition and lead to collapse of trading. (Dang, Gorton and Holmstrom 2011) model the agents’ information acquisition behavior through the standard costly state verification approach introduced by (Robert M. Townsend 1979). The agents can decide whether to pay a fixed cost to observe the exact value of the underlying asset or observe nothing. Therefore, information acquisition is endogenously determined. However, agents are only allowed to purchase an information structure\(^3\) with either all information or none. These two types of information structure are extreme. In particular, it is implicitly assumed that all intermediate information structures are assigned an infinite cost. This rigidity leads to infinitely many optimal securities, including a standard debt contract. All the other optimal securities look like the optimal debt contract but do not have the flat part. This non-uniqueness\(^4\) result is undesirable since it weakens their argument that debt is essential in financial crises.

(Dang, Gorton and Holmstrom 2011) also address the securitization process. They conduct an comparison between a debt portfolio (i.e., a single debt backed by all cash flows) and a portfolio of debts that each is backed by an individual cash flow. They numerically show that the former is less information-sensitive than the latter when underlying cash flows are independently and uniformly distributed on the unit interval. This result partly rationalizes the securitization process, but it is incomplete for three reasons. First, the comparison only reveals the dominance of debt portfolio over portfolio of debts. They do not make a comparison with other ABSs which would be required to establish the full optimality of debt. Second, the analysis only deals with independent cash flows, but in practice there often exists complex stochastic interdependence among underlying assets. Third, the argument relies on numerical solutions. Since the distributional details of underlying cash flows matter a lot, it seems unlikely that the above problems could be solved within any rigid information acquisition framework. Our model provides an alternative approach that pins down all these difficulties at the same time.

The point of departure of this paper relative to the security design literature is its endogenous and flexible information structure. Endogeneity refers to that the agents can choose from a set of information structures. Information acquisition is flexible in the sense that the choice set of information structures consists of all conditional distributions of signals given the state of the world. Therefore, our model differs from (Dang, Gorton and Holmstrom 2011) in the aspect of restrictions imposed on information acquisition. This flexibility plays a key role in our analysis. In contrast to the nothing-or-all information acquisition in (Dang, Gorton and Holmstrom 2011), our agents are allowed to gradually acquire information that helps distinguish any states with different payoffs. Since information acquisition generates adverse selection that is harmful to the seller,

\(^3\)An information structure is characterized by a conditional distribution of signals given the state of the world.

\(^4\)(Dang, Gorton and Holmstrom 2011) regain the unique optimality of debt by introducing public interim information and showing that debt is the least interim information-sensitive. But the argument relies on an infinite informational cost and specific distributional conditions.
she maintains a constant repayment level whenever it is off the limited liability boundary. When the cash flows are too low to support this level, the ABS reaches the boundary and thus equals the sum of underlying cash flows. Hence in contrast to (Dang, Gorton and Holmstrom 2011), our optimal security always consists of a boundary part and a flat part. We do not have to deal with the non-uniqueness. Moreover, pooling is naturally derived from the limited liability boundary, regardless of any distributional details of the underlying cash flows. Therefore, issuing securitized debt proved to be uniquely optimal within the set of all feasible ABSs.

Our qualitative results are also robust to the allocation of bargaining power. If instead the buyer designs the contract and proposes a take-it-or-leave-it offer to the seller, the optimal ABS is still a securitized debt, although the face value and price may change. We proceed as following. Section I studies endogenous and flexible information acquisition in a binary choice problem. The result is employed in Section II to derive the optimal ABS that provides liquidity. We discuss and conclude in Section III.

I. Binary Choice with Endogenous and Flexible Information Acquisition

Before introducing the economic environment, we review the logic of binary choice with endogenous and flexible information acquisition, which will play a key role in the following analysis.

In our leading example mentioned in the introduction, a buyer faces a take-it-or-leave-it offer. She has to acquire information and then make a binary choice. We first focus on information structures with binary signals and then show that it suffices to do so.

A. Decision Problem

Consider an agent who has to choose an action \( a \in \{0, 1\} \) and will receive a payoff \( u(a, \theta) \), where \( \theta \in \Theta \subset \mathbb{R} \) is an unknown state distributed according to a full support probability measure \( P \) over \( \Theta \).

The agent has access to a set of binary-signal information structures. In particular, she observes signals \( x \in \{0, 1\} \) parameterized by measurable function \( m : \Theta \to [0, 1] \), where \( m(\theta) \) is the probability of observing signal 1 if the true state is \( \theta \) (and so \( 1 - m(\theta) \) is the probability of observing signal 0). The conditional probability function \( m(\theta) \) describes the agent’s information acquisition strategy. By choosing different functional forms for \( m(\theta) \), the agent can make her signal covary with fundamental in any way she would like. Intuitively, if her welfare is sensitive to fluctuation of the state within some range \( A \subset \Theta \), she would pay much attention to this event by letting \( m(\theta) \) be highly sensitive to \( \theta \in A \). In this sense, choosing an information structure can be considered as hiring an analyst to write a report with emphasis on your interests.

Write \( M \triangleq \{m \in L(\Theta, P) : \forall \theta \in \Theta, m(\theta) \in [0, 1]\} \) for the set of binary-signal information structures. Let \( c : M \to \mathbb{R}_+ \) be the cost (in terms of utility) of acquiring information. We assume that

\[
(1) \quad c(m) = \mu \cdot \left[ \int_{\Theta} g(m(\theta)) dP(\theta) - g \left( \int_{\Theta} m(\theta) dP(\theta) \right) \right],
\]
where $\mu > 0$ and 
\[
g(x) = x \cdot \ln x + (1 - x) \cdot \ln (1 - x) .
\]

According to Shannon’s information theory, $\mu^{-1} \cdot c(m)$ is the mutual information between signal $x$ and state $\theta$.\(^5\) It measures the amount of information conveyed by information structure $m$ in the sense that observing signal $x$ reduces the agent’s subjective uncertainty (i.e., entropy) of $\theta$ by a magnitude of $\mu^{-1} \cdot c(m)$. $\mu > 0$ measures the difficulty in acquiring information. When $\mu = 0$, information acquisition incurs no cost and the agent can directly observe the true state. When $\mu \to \infty$, the agent cannot acquire any information at all. Note that $m(\theta) \equiv \text{constant}$ is a trivial information structure that conveys no information about $\theta$. It is straightforward to verify this by showing $c(\text{constant}) = 0$. It is also worth noting that $c(\cdot)$ is convex, i.e.,
\[
c(t \cdot m_1 + (1 - t) \cdot m_2) \leq t \cdot c(m_1) + (1 - t) \cdot c(m_2)
\]
for all $m_1, m_2 \in M$ and $t \in [0, 1]$. This convexity is strict when at least one of $m_1$ and $m_2$ is not a constant in $\theta$.

Now we are interested in the problem of an agent choosing an information structure $m \in M$ and a stochastic decision rule $f : [0, 1] \rightarrow [0, 1]$ to maximize her expected utility
\[(2)\]
\[
V(m, f) = \int_\Theta \left\{ + m(\theta) f(1) + (1 - m(\theta)) f(0) \right\} \cdot u(1, \theta) + m(\theta) (1 - f(1)) + (1 - m(\theta)) (1 - f(0)) \right\} \cdot u(0, \theta) \right\} dP(\theta) - c(m) .
\]

Without loss of generality, we can let $f = f^*$ where $f^*(1) = 1$ and $f^*(0) = 0$. This simplification is based on the following observation. If we let
\[
m^*(\theta) = m(\theta) f(1) + (1 - m(\theta)) f(0) ,
\]
then $V(m^*, f^*) \geq V(m, f)$, since the first term of (2) remains the same, while, by the convexity of $c(\cdot)$, the information cost becomes smaller.

Fixing $f = f^*$, we can interpret $m$ as a joint information structure and decision rule specifying that the agent will take action 1 with probability $m(\theta)$ in state $\theta$.

Now the agent’s problem is to choose $m \in M$ to maximize
\[
V^*(m) = \int_\Theta m(\theta) u(1, \theta) + [1 - m(\theta)] u(0, \theta) dP(\theta) - c(m) = \int_\Theta m(\theta) [u(1, \theta) - u(0, \theta)] dP(\theta) - c(m) + \int_\Theta u(0, \theta) dP(\theta) .
\]

Since $\int_\Theta u(0, \theta) dP(\theta)$ is a constant that does not depend on $m$, we can redefine the

\(^5\)Following the convention of information theory, we let $0 \cdot \ln 0 = 0$. This is reasonable since $\lim_{x \to 0} x \cdot \ln x = 0$.

\(^6\)This is essentially the unique measure of information given three axioms. See (Thomas M. Cover and Joy A. Thomas 1991) for more details.
agent’s objective as
\[
\max_{m \in M} V^* (m) = \int_{\Theta} \Delta u (\theta) \cdot m (\theta) \, dP (\theta) - c (m),
\]
where
\[
\Delta u (\theta) = u (1, \theta) - u (0, \theta)
\]
is the payoff gain from taking action 1 over action 0.

Without loss of generality, let \( \Pr (\Delta u (\theta) \neq 0) > 0 \). Then the following lemma characterizes the optimal strategy \( m \) for the agent.\(^7\)

**Proposition 1:** \(^8\)Let \( \Pr (\Delta u (\theta) \neq 0) > 0 \) to exclude the trivial case that the agent is always indifferent between the two actions. Let \( m \in M \) be an optimal strategy and
\[
p_1 = \int_{\Theta} m (\theta) \, dP (\theta)
\]
be the corresponding unconditional probability of taking action 1. Then,

i) the optimal strategy is unique;

ii) the optimal strategy is an interior point of \( M \) (i.e., \( p_1 \in (0, 1) \)) if and only if
\[
\int_{\Theta} \exp (\mu^{-1} \Delta u (\theta)) \, dP (\theta) > 1 \quad \text{and} \quad \int_{\Theta} \exp (-\mu^{-1} \Delta u (\theta)) \, dP (\theta) > 1;
\]
in this case, the optimal strategy \( m \) is characterized by
\[
\Delta u (\theta) = \mu \cdot \left[ g' (m (\theta)) - g' (p_1) \right]
\]
for all \( \theta \in \Theta \);

iii) the optimal strategy is \( p_1 = 1 \) (i.e., \( m (\theta) = 1 \) a.s.) if and only if
\[
\int_{\Theta} \exp (\mu^{-1} \Delta u (\theta)) \, dP (\theta) > 1 \quad \text{and} \quad \int_{\Theta} \exp (-\mu^{-1} \Delta u (\theta)) \, dP (\theta) \leq 1;
\]

iv) the optimal strategy is \( p_1 = 0 \) (i.e., \( m (\theta) = 0 \) a.s.) if and only if
\[
\int_{\Theta} \exp (\mu^{-1} \Delta u (\theta)) \, dP (\theta) \leq 1 \quad \text{and} \quad \int_{\Theta} \exp (-\mu^{-1} \Delta u (\theta)) \, dP (\theta) > 1;
\]

v) the three cases specified by (3), (5) and (6) exhaust all possibilities.

**Proof:** See Appendix A.

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\(^7\)We became aware of the related work (Michael Woodford 2008) while working on this paper. Here we use Lemma 2 of (Woodford 2008) to characterize the optimal strategy. To maintain the completeness of our paper, we give a proof in our context.

\(^8\)We do not have to require \( \Theta \subset \mathbb{R} \). This proposition holds for any probability space \( \Theta \).
These results are intuitive. Since the information cost is convex, the agent’s objective is concave, which gives rise to the uniqueness of the optimal strategy.

Roughly speaking, condition (5) means that the payoff from taking action 1 exceeds that from action 0 "on average". Thus the agent would like to always take action 1 without acquiring information. An extreme case under this condition is that when action 1 is dominant, i.e., the payoff gain $\Delta u (\theta) > 0$ almost surely. It is obvious that the agent will always take action 1 regardless of $\mu$, the marginal cost of information acquisition.

When neither action is dominant, i.e.,

$$\text{Pr} (\Delta u (\theta) > 0) > 0 \quad \text{and} \quad \text{Pr} (\Delta u (\theta) < 0) > 0,$$

the marginal cost of information acquisition $\mu$ plays a role. On the one hand,

$$\lim_{\mu \to \infty} \int \exp (\pm \mu^{-1} \Delta u (\theta)) \, dP (\theta) = 1.$$  

Thus Proposition 1 predicts that no information is acquired if $\mu$ is high enough. On the other hand, since

$$\lim_{\mu \to 0} \frac{d}{d\mu^{-1}} \int \exp (\mu^{-1} \Delta u (\theta)) \, dP (\theta) =$$

$$= \lim_{\mu \to 0} \int \exp (\mu^{-1} \Delta u (\theta)) \Delta u (\theta) \, dP (\theta)$$

$$= \lim_{\mu \to 0} \int_{\Delta u (\theta) > 0} \exp (\mu^{-1} \Delta u (\theta)) \Delta u (\theta) \, dP (\theta)$$

$$+ \text{Pr} (\Delta u (\theta) = 0) + \lim_{\mu \to 0} \int_{\Delta u (\theta) < 0} \exp (\mu^{-1} \Delta u (\theta)) \Delta u (\theta) \, dP (\theta)$$

$$= +\infty + \text{Pr} (\Delta u (\theta) = 0) + 0$$

$$= +\infty,$$

we have

$$\lim_{\mu \to 0} \int \exp (\mu^{-1} \Delta u (\theta)) \, dP (\theta) = 1.$$  

A similar argument leads to

$$\lim_{\mu \to 0} \int \exp (-\mu^{-1} \Delta u (\theta)) \, dP (\theta) = 1.$$  

Therefore, Proposition 1 reads that there must be information acquisition if the marginal cost of information is sufficiently low. This interpretation coincides our intuition that the agent rationally decides whether to acquire information through comparing the cost to the benefit of information acquisition. Condition (6) can be interpreted in a similar way.

When neither action is dominant and the marginal cost of information acquisition takes intermediate values, the agent finds it optimal to acquire some information to make her
action (partially) contingent on the fundamentals. This is the case specified by condition (3). Since \( g' \) is strictly increasing, (4) implies that \( m(\theta) \), the conditional probability of choosing action 1, is increasing with respect to payoff gain \( \Delta u(\theta) \). This is intuitive. The left hand side of (4) represents the marginal benefit of increasing \( m(\theta) \), while the right hand side of (4) is the marginal cost of information when increasing \( m(\theta) \). Therefore, if deciding to acquire information, the agent will equate her marginal benefit with her marginal cost of doing so.

**AN EXAMPLE**

The following example provides some intuition behind the agent’s information acquisition strategy.

Let \( \theta \) be distributed according to \( N(t, 1) \) and

\[
\Delta u(\theta) = \theta.
\]

It is easy to verify that the agent always chooses action 1 (action 0) if and only if \( t \geq \mu^{-1}/2 \) (\( t \leq -\mu^{-1}/2 \)). In this case, action 1 (action 0) is superior to action 0 (action 1) with high probability (i.e., \( t \) is large) and the difficulty in acquiring information is relatively high (i.e., \( \mu \) is large). Thus it is not worth acquiring any information at all.

Let \( t = 0 \), then the agent finds it optimal to acquire some information. According to (4), the optimal information acquisition strategy \( m(\theta) \) satisfies

\[
\theta/\mu = g'(m(\theta)) - g' \left( \int_\theta m(\theta) dP(\theta) \right),
\]

where

\[
g'(m) = \ln \frac{m}{1-m}.
\]

Since prior \( N(0, 1) \) is symmetric about the origin and payoff gain \( \Delta u(\theta) \) is an odd function, the agent is indifferent on average, i.e., \( \int_\theta m(\theta) dP(\theta) = 1/2 \). Thus \( g' \left( \int_\theta m(\theta) dP(\theta) \right) = 0 \) and (7) becomes

\[
\theta/\mu = \ln \frac{m(\theta)}{1-m(\theta)}.
\]

Therefore,

\[
m(\theta) = \frac{1}{1 + \exp (-\theta/\mu)}.
\]

First note that

\[
\lim_{\mu \to 0} m(\theta) = a(\theta) \triangleq \begin{cases} 1 & \text{if } \theta \geq 0 \\ 0 & \text{if } \theta < 0 \end{cases}.
\]

Step function \( a(\theta) \) is the agent’s best response when \( \mu = 0 \). In this case, the agent can observe the exact value of \( \theta \). When \( \mu > 0 \), the best response is characterized by (8). Since information is no longer free, the agent has to allow some mistakes in her response.
The conditional probability of mistake is given by

$$|m(\theta) - a(\theta)|,$$

which is decreasing in $|\theta|$, the cost of mistake. Therefore, the agent deliberately acquires information to balance the cost of mistake and the cost of information.

Second, parameter $\mu$ measures the difficulty in acquiring information. Figure 1 shows how $m(\theta)$ varies with this parameter.

![Figure 1. Information acquisition under various information costs](image)

When $\mu = 0$, information acquisition incurs no cost and the agent’s response is a step function. She never makes mistake. When $\mu$ becomes larger, she starts to compromise the accuracy of her decision to save information cost. Larger $\mu$ leads to flatter $m(\theta)$. Finally, when $\mu$ is extremely large, $m(\theta)$ is almost constant and the agent almost stops acquiring information.

Third, since the agent’s action is highly sensitive to $\theta$ where slope $\left| \frac{dm(\theta)}{d\theta} \right|$ is large, $\left| \frac{dm(\theta)}{d\theta} \right|$ reflects her attentiveness around $\theta$. Under this interpretation, Figure 1 reveals that the agent actively collects information for intermediate values of state but is rationally
inattentive to values at the tails. This result coincides our intuition. The agent does not care about whether her payoff is five hundred or six hundred dollars, since she should choose action 1 in either case. But she does care about if it is positive or not. Therefore she pay attention to the region around zero.

We have been focusing on binary-signal information structures. Next subsection justifies this setup.

B. Justifying the Binary-signal Information Structure

Generally, an agent can purchase any information structure \((X, \sigma, \pi)\). Here \(X\) is the set of realizations of the agent’s signal, \(\sigma\) is a \(\sigma\)-algebra on \(X\), and \(\forall \theta \in \Theta, \pi \left( \cdot \mid \theta \right)\) is a probability measure on \(X\). \(\pi \left( \cdot \mid \theta \right)\) conveys information about state \(\theta\) in the sense that for any event \(A \subset X\), \(\pi \left( A \mid \theta \right)\) specifies the conditional probability of \(A\) given \(\theta\). Before making a decision, the agent can acquire information about the state in the form of an information structure. An information structure specifies how much and what kind of information to acquire.

The binary-signal information structure analyzed in last subsection is a special case with \(X = \{0, 1\}\) and \(\pi \left( 1 \mid \theta \right) = m \left( \theta \right)\) (and so \(\pi \left( 0 \mid \theta \right) = 1 - m \left( \theta \right)\)). For binary choice problem with information acquisition, it is sufficient to restrict our attention to this special class of information structures. To see this, let \(((X, \sigma), \pi)\) be any information acquisition strategy of the agent. Given this information structure, the agent optimally chooses her action rule as \(a : X \rightarrow [0, 1]\), where \(a(x)\) is the probability of taking action 1 upon receiving signal \(x\). Let

\[
X_1 = \{ x \in X : a(x) = 1 \},
\]

\[
X_0 = \{ x \in X : a(x) = 0 \},
\]

and

\[
X_{ind} = \{ x \in X : a(x) \in (0, 1) \}.
\]

\(X_1 (X_0)\) is the set of signal realizations such that the agent definitely takes action 1 (0). She is indifferent when her signal belongs to \(X_{ind}\). Then \((X_1, X_0, X_{ind})\) forms a partition of \(X\). It is worth noting that the agent has no incentive to discern signal realizations within any of \(X_1, X_0\) and \(X_{ind}\), since this effort requires more information but generates no extra benefit. In addition, because she is indifferent between action 0 and 1 upon event \(X_{ind}\), she would rationally pay no attention to distinguish this event from other realizations. Hence, the agent always play pure strategies upon receiving her signal. Therefore, the agent always prefers binary-signal information structures. (Woodford 2009) has an similar argument that the agent only needs to acquire a "yes/no" signal. A detailed constructive proof can be found in Lemma 1 of (Yang 2011).
II. Security Design with Information Acquisition

A. Basic Setup

We consider a two-period game with two players. One player is a seller that owns \( N \) assets at period 0. These assets generate verifiable random cash flows \( \theta \in \Theta \subseteq \mathbb{R}_+^N \) in period 0.\(^9\) The other player is a potential buyer holding consumption goods (money) at period 0. Player \( i \)'s utility function is given by

\[
(9) \quad u_i = c_i0 + \delta_i \cdot c_i1,
\]

where \( c_{it} \) denotes player \( i \)'s consumption at period \( t \) and \( \delta_i \in [0, 1] \) is her subjective discount factor, \( i \in \{s, b\} \) (\( \{s, b\} \) stands for \{seller, buyer\}). We assume \( \delta_b > \delta_s \) to represent that the seller has more profitable project to take than do the buyer. This assumption creates the trading demand. Both agents may benefit from transferring some goods to the seller at date 0 and compensating the buyer with repayment backed by the random cash flows \( \theta \) at date 1.

The two agents start with identical information about \( \theta \), which is represented by a full support common prior \( P \) over \( \Theta \). Without loss of generality, we assume that \( P \) is absolutely continuous with respect to Lebesgue’s measure on \( \mathbb{R}_+^N \).

A security backed by \( \theta \), the cash flows of the \( N \) assets, is a mapping \( s : \Theta \rightarrow \mathbb{R} \) such that \( \forall \theta \in \Theta, s(\theta) \in [0, \sum_{n=1}^N \theta_n] \). A contract \( (s(\cdot), q) \) is a security \( s(\cdot) \) associated with a price \( q > 0 \). Throughout the paper, we focus on the case where one player proposes a take-it-or-leave-it contract \( (s(\cdot), q) \) to her opponent, who then acquires information and decides whether to accept it. This setup captures the idea that some agents in the markets of securitized assets are less sophisticated than others and cannot produce private information about the underlying cash flows. This setup also makes our problem tractable. We would have to study a much more complicated signaling game if the issuer can produce private information before her proposal. In that case, the set of possible signals consists of all contracts, which is a functional space. To the best of our knowledge, this kind of signaling games are rarely studied before.\(^10\) In the literature, either the informed agent chooses finite-dimension signals (e.g. the level of debt in (Stephen A. Ross 1977), the retaining fraction of the equity in (Hayne E Leland and David H Pyle 1977), etc.), or the issuer designs the security before her information acquisition (e.g. (DeMarzo and Duffie 1999), (Biais and Mariotti 2005)).

B. Optimal Contract when the Seller Designs

Consider the particular binary choice problem where the agent is a risk neutral buyer with utility (9), and action 1 corresponds to buying the ABS with return \( s(\theta) \) at price

\(^9\)Here the assumption of verifiable cash flows is natural, since we generally have third parties monitor and collect the underlying loans and distribute the cash flows to the holders of asset backed securities.

\(^{10}\)(DeMarzo, Kremer and Skrzypacz 2005) does consider a security design problem where potential signals are securities. But their approach does not fit our framework of endogenous information acquisition.
$q$ and action 0 corresponds to not buying. Write $m_{s,q}$ for the buyer’s optimal strategy when facing $(s,q)$. Let

$$ \overline{p}_{s,q} = \int_{\Theta} m_{s,q}(\theta) \, dP(\theta) $$

be the buyer’s unconditional probability of accepting the offer. The uninformed seller thus enjoys expected utility

$$ W(s,q) = \int_{\Theta} m_{s,q}(\theta) \cdot \left[ q - \delta_s \cdot s(\theta) \right] \, dP(\theta). $$

The seller’s problem is to choose a feasible contract $(s,q)$ satisfying $s(\theta) \cdot s(\theta) \leq 0$ to maximize $W(s,q)$. Let $(s^*(\cdot), q^*)$ denote the optimal contract and

$$ \overline{p}_{s^*,q^*} = \int_{\Theta} m_{s^*,q^*}(\theta) \, dP(\theta) $$

be the corresponding probability of trade.

According to Proposition 1, there are three possible cases: i) $\overline{p}_{s^*,q^*} = 0$; ii) $\overline{p}_{s^*,q^*} = 1$; and iii) $\overline{p}_{s^*,q^*} \in (0, 1)$. We first argue that case i) is impossible.

**PROPOSITION 2:** $\overline{p}_{s^*,q^*} > 0$, i.e., trade happens with positive probability.

**PROOF:** Let $\beta \in (\delta_s^{-1} \delta_b^{-1}, 1)$ and

$$ f(q) = \int_{\Theta} \min\left( \sum_{n=1}^{N} \theta_n, \beta \delta_s^{-1} q \right) \, dP(\theta). $$

Since $P$ is a continuous distribution and $\beta^{-1} \delta_s \delta_b^{-1} < 1$, there exists $q_0 > 0$ s.t.

$$ \Pr\left( \sum_{n=1}^{N} \theta_n \geq \beta \delta_s^{-1} q \right) > \beta^{-1} \delta_s \delta_b^{-1} $$

for all $q \in [0, q_0]$. Thus for any $q \in (0, q_0)$,

$$ f'(q) = \beta \delta_s^{-1} \int_{\Theta} \min\left[ \theta \in \Theta : \sum_{n=1}^{N} \theta_n \geq \beta \delta_s^{-1} q \right] \cdot dP(\theta) $$

$$ = \Pr\left( \sum_{n=1}^{N} \theta_n \geq \beta \delta_s^{-1} q \right) \cdot \beta \delta_s^{-1} $$

$$ > \beta^{-1} \delta_s \delta_b^{-1} \cdot \beta \delta_s^{-1} = \delta_b^{-1}. $$

Note that

$$ f(0) = 0. $$
which implies that
\[ f(q) > \delta^{-1}q \]
for all \( q \in (0, q_0) \).

Consider a securitized debt
\[ s(\theta) = \min \left( \sum_{n=1}^{N} \theta_n, D \right) \]
with face value \( D = \beta \delta^{-1}q \) and price \( q \in (0, q_0) \). The buyer’s payoff gain from accepting this offer over rejecting it is
\[ \Delta u(\theta) = \delta_s \cdot s(\theta) - q. \]

By Jensen’s inequality,
\[
\int_\Theta \exp \left( \mu^{-1} \Delta u(\theta) \right) dP(\theta) \\
\geq \exp \left( \mu^{-1} \int_\Theta \Delta u(\theta) dP(\theta) \right) \\
= \exp \left( \mu^{-1} \left[ \delta_b \cdot \int_\Theta \min \left( \sum_{n=1}^{N} \theta_n, \beta \delta^{-1}q \right) dP(\theta) - q \right] \right) \\
= \exp \left( \mu^{-1} \left[ \delta_b \cdot f(q) - q \right] \right) \\
> \exp(0) = 1,
\]
where the last inequality comes from (11). Thus according to Proposition 1, \( \overline{p}_{s,q} > 0 \). Then, the seller’s expected payoff from this contract is
\[
W(s, q) = \int_\Theta m_{s,q}(\theta) \cdot \left[ q - \delta_s \cdot s(\theta) \right] dP(\theta) \\
= \int_\Theta m_{s,q}(\theta) \left[ q - \delta_s \cdot \min \left( \sum_{n=1}^{N} \theta_n, \beta \delta^{-1}q \right) \right] dP(\theta) \\
\geq \int_\Theta m_{s,q}(\theta) \left[ q - \delta_s \cdot \beta \delta_s^{-1}q \right] dP(\theta) \\
= (1 - \beta)q \cdot \overline{p}_{s,q} > 0.
\]

By definition, the seller’s expected payoff through the optimal contract is \( W(s^*, q^*) \geq W(s, q) > 0 \). This directly implies \( \overline{p}_{s^*,q^*} > 0 \) since \( \overline{p}_{s^*,q^*} = 0 \) always generates zero expected payoff to the seller.

According to Proposition 2, the seller always prefers trade although facing adverse selection. This is because the seller owns all bargaining power. She is able to minimize the negative effect of information acquisition through designing an appropriate contract
and thus enjoy the benefit from trading.

PROPOSITION 3: If

\[ D_0 \triangleq \inf \left\{ \sum_{n=1}^{N} \theta_n : \tilde{\theta} \in \Theta \right\} > 0, \]

the seller finds it optimal to first issue a riskless debt with face value \( D_0 \) and price \( \delta_b D_0 \), and then sell an ABS backed by the residual cash flows.

PROOF: Let \((s^* (\cdot), q^*)\) be the optimal contract written on the whole cash flows. First note that \( s^* (\tilde{\theta}) \geq D_0 \) almost surely and \( q^* \geq \delta_b D_0 \). Let \( m_{s^*, q^*} (\tilde{\theta}) \) be the buyer’s best response to \((s^* (\cdot), q^*)\). Then \( m_{s^*, q^*} (\tilde{\theta}) \) is also the best response to contract \((s^* (\cdot) - D_0, q^* - \delta_b D_0)\) since the two contracts have the same payoff gain. The buyer’s payoff from purchasing the riskless debt \( D_0 \) at price \( \delta_b D_0 \) is zero. Without loss of generality, we assume that the buyer always purchase this debt.\(^\text{11}\) The seller’s expected payoff from issuing a riskless debt \( D_0 \) and selling the contract \((s^* (\cdot) - D_0, q^* - \delta_b D_0)\) is

\[
\delta_b D_0 - \delta_s D_0 + \int_{\Theta} m_{s^*, q^*} (\tilde{\theta}) \cdot \left[ (q^* - \delta_b D_0) - \delta_s \cdot \left( s^* \left( \tilde{\theta} \right) - D_0 \right) \right] d P (\tilde{\theta})
\]

\[
= (\delta_b - \delta_s) D_0 + \int_{\Theta} m_{s^*, q^*} (\tilde{\theta}) \cdot \left[ q^* - \delta_s \cdot s^* \left( \tilde{\theta} \right) \right] d P (\tilde{\theta}) - (\delta_b - \delta_s) D_0 \cdot \int_{\Theta} m_{s^*, q^*} (\tilde{\theta}) d P (\tilde{\theta})
\]

\[
= \int_{\Theta} m_{s^*, q^*} (\tilde{\theta}) \cdot \left[ q^* - \delta_s \cdot s^* \left( \tilde{\theta} \right) \right] d P (\tilde{\theta}) + (1 - \overline{P}_{s^*, q^*}) (\delta_b - \delta_s) D_0
\]

\[
= W (s^*, q^*) + (1 - \overline{P}_{s^*, q^*}) (\delta_b - \delta_s) D_0 \geq W (s^*, q^*) .
\]

If \( \overline{P}_{s^*, q^*} < 1 \), the above inequality holds strictly. Thus issuing the riskless debt \( D_0 \) at price \( \delta_b D_0 \) and selling the contract \((s^* (\cdot) - D_0, q^* - \delta_b D_0)\) strictly dominates selling \((s^* (\cdot), q^*)\) directly. If \( \overline{P}_{s^*, q^*} = 1 \), the optimal contract \((s^* (\cdot), q^*)\) is just the riskless debt \( D_0 \) with price \( \delta_b D_0 \), which is certainly accepted by the buyer without information acquisition. After selling this riskless debt, the seller can issue an ABS backed by the residual cash flows. This makes the seller strictly better off than just issuing the riskless debt according to Proposition 2. This concludes the proof.

According to Proposition 3, the seller should always design an ABS after issuing the riskless component of the underlying cash flows. Without loss of generality, we can assume

\[
\inf \left\{ \sum_{n=1}^{N} \theta_n : \tilde{\theta} \in \Theta \right\} = 0
\]

and focus on the security design problem.

\(^{11}\)This can be achieved by reducing the price a tiny bit.
According to Proposition 2, only case ii) and iii) are possible. In case ii) \( P^s_{x^*,q^*} = 1 \) and the buyer does not acquire any information. In case iii), \( P^s_{x^*,q^*} \in (0, 1) \) and the buyer does acquire some information. The next two subsections characterize the optimal contracts in both cases, respectively.

**The Optimal Contract without Inducing Information Acquisition**

**Proposition 4:** If the seller’s optimal contract induces the buyer to always accept it without acquiring information, it must be a securitized debt

\[
s^* (\theta) = \min \left( \sum_{n=1}^{N} \theta_n, D^* \right)
\]

with price \( q^* \), where

\[
D^* = \mu \delta_b^{-1} \cdot [\ln \delta_b - \ln \delta_s] + \delta_b^{-1} q^* ,
\]

\( q^* \) is the unique fixed point of

\[
h (q) = -\mu \ln E \exp \left( -\mu^{-1} \delta_b \cdot \min \left( \sum_{n=1}^{N} \theta_n, \mu \delta_b^{-1} \cdot [\ln \delta_b - \ln \delta_s] + \delta_b^{-1} q \right) \right)
\]

and the expectation is taken according to common prior \( P \).

**Proof:** According to Proposition 2, the seller chooses an \( (s (\cdot), q) \) to maximize

\[
\int_{\Theta} \left[ q - s (\theta) \right] dP (\theta)
\]

subject to

\[
E \exp \left( \mu^{-1} \left[ \delta_b \cdot s (\theta) - q \right] \right) > 1 ,
\]

\[
E \exp \left( -\mu^{-1} \left[ \delta_b \cdot s (\theta) - q \right] \right) \leq 1
\]

and

\[
s (\theta) \in \left[ 0, \sum_{n=1}^{N} \theta_n \right] .
\]

First note that condition (14) cannot bind. Otherwise condition (15) must be violated according to Jensen’s inequality. Second, condition (15) must bind for the optimal contract, otherwise the seller can always benefit from increasing the price \( q \). Moreover,
when condition (15) binds, Jensen’s inequality implies condition (14). Thus, condition (14) and (15) reduce to
\[ E \exp \left( -\mu^{-1} \left[ \delta_b \cdot s \left( \bar{\theta} \right) - q \right] \right) = 1 , \]
i.e.,
(17)
\[ q = -\mu \ln E \exp \left( -\mu^{-1} \delta_b \cdot s \left( \bar{\theta} \right) \right) . \]

Plugging (17) into the seller’s objective function (13), we can rewrite the seller’s problem as
\[
\min_{s(\cdot)} \mu \ln E \exp \left( -\mu^{-1} \delta_b \cdot s \left( \bar{\theta} \right) \right) + \delta_s \cdot E s \left( \bar{\theta} \right)
\]
subject to (16).

Let \( s \left( \bar{\theta} \right) = s^* \left( \bar{\theta} \right) + \alpha \cdot \varepsilon \left( \bar{\theta} \right) \) be an arbitrary perturbation of the optimal security \( s^* \left( \cdot \right) \). Let
\[ J \left( \alpha \right) = \mu \ln E \exp \left( -\mu^{-1} \delta_b \cdot s \left( \bar{\theta} \right) \right) + \delta_s \cdot E s \left( \bar{\theta} \right) . \]

Taking first order variation leads to
\[
\frac{dJ}{d\alpha} \bigg|_{\alpha=0} = -\delta_b \left[ E \exp \left( -\mu^{-1} \delta_b \cdot s^* \left( \bar{\theta} \right) \right) \right]^{-1} E \left[ \exp \left( -\mu^{-1} \delta_b \cdot s^* \left( \bar{\theta} \right) \right) \cdot \varepsilon \left( \bar{\theta} \right) \right] + \delta_s \cdot E \varepsilon \left( \bar{\theta} \right)
\]
\[ = E \left[ \left( \delta_s - \delta_b \right) E \left[ \exp \left( -\mu^{-1} \delta_b \cdot s^* \left( \bar{\theta} \right) \right) \right]^{-1} \exp \left( -\mu^{-1} \delta_b \cdot s^* \left( \bar{\theta} \right) \right) \cdot \varepsilon \left( \bar{\theta} \right) \right] \]
\[ = E \left[ \delta_s \varepsilon \left( \bar{\theta} \right) \cdot \varepsilon \left( \bar{\theta} \right) \right] . \]

Let
\[ A_0 = \left\{ \bar{\theta} \in \Theta : \bar{\theta} \neq \bar{0} , s^* \left( \bar{\theta} \right) = 0 \right\} , \]
\[ A_1 = \left\{ \bar{\theta} \in \Theta : \bar{\theta} \neq \bar{0} , s^* \left( \bar{\theta} \right) \in \left( 0 , \sum_{n=1}^N \theta_n \right) \right\} \]
and
\[ A_2 = \left\{ \bar{\theta} \in \Theta : \bar{\theta} \neq \bar{0} , s^* \left( \bar{\theta} \right) = \sum_{n=1}^N \theta_n \right\} . \]

Then \( \{ A_0 , A_1 , A_2 \} \) is a partition of \( \Theta \setminus \left\{ \bar{0} \right\} \). Since \( s^* \left( \cdot \right) \) is the optimal security,
\[ \frac{dJ}{d\alpha} \bigg|_{\alpha=0} \geq 0 \]
holds for any feasible perturbation \( \epsilon \left( \tilde{\theta} \right) \). Hence, we have

\[
(19) \quad r \left( \tilde{\theta} \right) =
\begin{cases} 
0 & \text{if } \tilde{\theta} \in A_0 \\
\geq 0 & \text{if } \tilde{\theta} \in A_1 \\
\leq 0 & \text{if } \tilde{\theta} \in A_2
\end{cases}
\]

For any \( \tilde{\theta}' \in A_0 \), (19) implies \( r \left( \tilde{\theta}' \right) \geq 0 \), i.e.,

\[
\delta_s \geq \delta_b \left[ E \exp \left( -\mu^{-1} \delta_b \cdot s^* \left( \tilde{\theta} \right) \right) \right]^{-1} \exp \left( -\mu^{-1} \delta_b \cdot \left( \tilde{\theta}' \right) \right)
\]

\[
= \delta_b \left[ E \exp \left( -\mu^{-1} \delta_b \cdot \left( \tilde{\theta} \right) \right) \right]^{-1} \exp \left( -\mu^{-1} \delta_b \cdot 0 \right)
\]

\[
= \delta_b \left[ E \exp \left( -\mu^{-1} \delta_b \cdot \left( \tilde{\theta} \right) \right) \right]^{-1} \exp \left( -\mu^{-1} \delta_b \cdot \left( \tilde{\theta}' \right) \right)
\]

\[
\ln \delta_s \geq \ln \delta_b - \ln E \exp \left( -\mu^{-1} \delta_b \cdot s^* \left( \tilde{\theta} \right) \right)
\]

\[
= \ln \delta_b + \mu^{-1} q^*
\]

where the last equality comes from (17). Thus,

\[
\mu^{-1} q^* \leq \ln \delta_s - \ln \delta_b < 0
\]

which is a contradiction. Therefore,

\[
(20) \quad \Pr \left( A_0 \right) = 0
\]

For any \( \tilde{\theta}' \in A_1 \), (19) implies \( r \left( \tilde{\theta}' \right) = 0 \), i.e.,

\[
\delta_s = \delta_b \left[ E \exp \left( -\mu^{-1} \delta_b \cdot s^* \left( \tilde{\theta} \right) \right) \right]^{-1} \exp \left( -\mu^{-1} \delta_b \cdot \left( \tilde{\theta}' \right) \right)
\]

\[
\ln \delta_s = \ln \delta_b - \ln E \exp \left( -\mu^{-1} \delta_b \cdot s^* \left( \tilde{\theta} \right) \right) - \mu^{-1} \delta_b \cdot s^* \left( \tilde{\theta}' \right)
\]

\[
= \ln \delta_b + \mu^{-1} q^* - \mu^{-1} \delta_b \cdot s^* \left( \tilde{\theta}' \right)
\]

where the last equality follows (17). Therefore,

\[
(21) \quad s^* \left( \tilde{\theta}' \right) = \mu \delta_b^{-1} \cdot \left[ \ln \delta_b - \ln \delta_s \right] + \delta_b^{-1} q^*
\]
is a constant for all $\overrightarrow{\theta} \in A_1$.

For any $\overrightarrow{\theta} \in A_2$, (19) implies $r \left( \overrightarrow{\theta} \right) \leq 0$, i.e.,
\[
\delta_s \leq \delta_b \left[ \mathbb{E} \exp \left( -\mu^{-1} \delta_b \cdot s^* \left( \overrightarrow{\theta} \right) \right) \right]^{-1} \exp \left( -\mu^{-1} \delta_b \cdot s^* \left( \overrightarrow{\theta} \right) \right)
\]
\[
= \delta_b \left[ \mathbb{E} \exp \left( -\mu^{-1} \delta_b \cdot s^* \left( \overrightarrow{\theta} \right) \right) \right]^{-1} \exp \left( -\mu^{-1} \delta_b \cdot \sum_{n=1}^{N} \theta'_n \right),
\]
i.e.,
\[
\ln \delta_s \leq \ln \delta_b - \ln \mathbb{E} \exp \left( -\mu^{-1} \delta_b \cdot s^* \left( \overrightarrow{\theta} \right) \right) - \mu^{-1} \delta_b \cdot \sum_{n=1}^{N} \theta'_n
\]
\[
= \ln \delta_b + \mu^{-1} q^* - \mu^{-1} \delta_b \cdot \sum_{n=1}^{N} \theta'_n,
\]
where the last equality comes from (17). Therefore,
\[
\sum_{n=1}^{N} \theta'_n \leq \mu \delta_b^{-1} \cdot [\ln \delta_b - \ln \delta_s] + \delta_b^{-1} q^*.
\]
(22)

Let
\[
D^* = \mu \delta_b^{-1} \cdot [\ln \delta_b - \ln \delta_s] + \delta_b^{-1} q^*.
\]
Then, (20), (21) and (22) imply that
\[
s^* \left( \overrightarrow{\theta} \right) = \min \left( \sum_{n=1}^{N} \theta_n, D^* \right),
\]
i.e., the optimal security must be a securitized debt.

Finally, let
\[
h \left( q \right) = -\mu \ln \mathbb{E} \exp \left( -\mu^{-1} \delta_b \cdot \min \left( \sum_{n=1}^{N} \theta_n, \mu \delta_b^{-1} \cdot [\ln \delta_b - \ln \delta_s] + \delta_b^{-1} q \right) \right).
\]
We show that $q^* > 0$ and it is the unique fixed point of $h \left( q \right)$. 
By (17), we have
\[ q^* = -\mu \ln E \exp \left( -\mu^{-1} \delta_b \cdot s^* \left( \theta \right) \right) \]
\[ = -\mu \ln E \exp \left( -\mu^{-1} \delta_b \cdot \min \left( \sum_{n=1}^{N} \theta_n, D^* \right) \right) \]
\[ = -\mu \ln E \exp \left( -\mu^{-1} \delta_b \cdot \min \left( \sum_{n=1}^{N} \theta_n, \mu \delta_b^{-1} \cdot \left[ \ln \delta_b - \ln \delta_s \right] + \delta_b^{-1} q^* \right) \right) \]
\[ = h(q^*) . \]

Thus \( q^* \) is a fixed point of \( h(q) \). First note that (12) implies \( h(0) > 0 \). Second note that

\[ h'(q) \]
\[ = \left[ E \exp \left( -\mu^{-1} \delta_b \cdot \min \left( \sum_{n=1}^{N} \theta_n, \mu \delta_b^{-1} \cdot \left[ \ln \delta_b - \ln \delta_s \right] + \delta_b^{-1} q \right) \right) \right]^{-1} \]
\[ \cdot E \left[ \exp \left( -\mu^{-1} \delta_b \cdot \min \left( \sum_{n=1}^{N} \theta_n, \mu \delta_b^{-1} \cdot \left[ \ln \delta_b - \ln \delta_s \right] + \delta_b^{-1} q \right) \right) \right] \cdot \left[ \sum_{n=1}^{N} \theta_n \cdot \mu \delta_b^{-1} \cdot \left[ \ln \delta_b - \ln \delta_s \right] + \delta_b^{-1} q \right] \]
\[ \leq \left[ E \exp \left( -\mu^{-1} \delta_b \cdot \min \left( \sum_{n=1}^{N} \theta_n, \mu \delta_b^{-1} \cdot \left[ \ln \delta_b - \ln \delta_s \right] + \delta_b^{-1} q \right) \right) \right]^{-1} \]
\[ \cdot E \left[ \exp \left( -\mu^{-1} \delta_b \cdot \min \left( \sum_{n=1}^{N} \theta_n, \mu \delta_b^{-1} \cdot \left[ \ln \delta_b - \ln \delta_s \right] + \delta_b^{-1} q \right) \right) \right] \cdot 1 \]
\[ = 1 \]

and

\[ \lim_{q \to \infty} h'(q) \]
\[ = \left[ E \exp \left( -\mu^{-1} \delta_b \cdot \sum_{n=1}^{N} \theta_n \right) \right]^{-1} \cdot E \left[ \exp \left( -\mu^{-1} \delta_b \cdot \sum_{n=1}^{N} \theta_n \right) \right] \cdot \lim_{q \to \infty} \left[ \sum_{n=1}^{N} \theta_n \cdot \mu \delta_b^{-1} \cdot \left[ \ln \delta_b - \ln \delta_s \right] + \delta_b^{-1} q \right] \]
\[ = \left[ E \exp \left( -\mu^{-1} \delta_b \cdot \sum_{n=1}^{N} \theta_n \right) \right]^{-1} \cdot E \left[ \exp \left( -\mu^{-1} \delta_b \cdot \sum_{n=1}^{N} \theta_n \right) \right] \cdot 0 \]
\[ = 0 . \]

Hence, \( h(q) \) has a unique fixed point \( q^* > 0 \). This concludes the proof.

A bit manipulation of (17) in the above proof leads to an interesting observation. (17)
implies
\[
q^* = -\mu \ln E \exp \left( -\mu^{-1} \delta_b \cdot s^* \left( \vec{\theta} \right) \right)
\leq -\mu \ln \left( \exp \left( -\mu^{-1} \delta_b \cdot E s^* \left( \vec{\theta} \right) \right) \right)
= \delta_b \cdot E s^* \left( \vec{\theta} \right),
\]
where the inequality follows Jensen’s inequality. We find that the optimal price \( q^* \) is strictly lower than the "fair" price \( \delta_b \cdot E s^* \left( \vec{\theta} \right) \) although no information asymmetry occurs. This is because the seller has to lower the price to bribe the buyer not to acquire information.

**The Optimal Contract with Information Acquisition**

This subsection shows that if the optimal contract turns out to induce information acquisition, it still must be a securitized debt.

According to Proposition 1, any contract \((s(\cdot), q)\) that induces the buyer to acquire information must satisfy

\[
E \exp \left( \mu^{-1} \left[ \delta_b \cdot s \left( \vec{\theta} \right) - q \right] \right) > 1
\]

and

\[
E \exp \left( -\mu^{-1} \left[ \delta_b \cdot s \left( \vec{\theta} \right) - q \right] \right) > 1,
\]

where the expectation is taken according to common prior \(P\).

Given such a contract, Proposition 1 predicts that the buyer’s optimal strategy \(m_{s,q}\) is uniquely characterized by

\[
\delta_b \cdot s \left( \vec{\theta} \right) - q = \mu \cdot \left[ g' \left( m_{s,q} \left( \vec{\theta} \right) \right) - g' \left( \vec{p}_{s,q} \right) \right],
\]

where

\[
\vec{p}_{s,q} = \int_{\Theta} m_{s,q} \left( \vec{\theta} \right) dP \left( \vec{\theta} \right)
\]
is the buyer’s unconditional probability of accepting the offer.

The seller chooses \((s(\cdot), q)\) to maximize her expected payoff

\[
W \left( s, q \right) = \int_{\Theta} m_{s,q} \left( \vec{\theta} \right) \cdot \left[ q - \delta_s \cdot s \left( \vec{\theta} \right) \right] dP \left( \vec{\theta} \right)
\]
subject to (23), (24), (25) and

\[(26) \quad s^* (\vec{\theta}) \in \left[ 0, \sum_{n=1}^{N} \theta_n \right], \]

where (26) is the feasibility condition.

Because of the strict inequalities, both (23) and (24) are not binding for the optimal contract. Thus conditional on the fact that the optimal contract satisfies (23) and (24), we do not have to take into account these two constraints in our analysis.

We derive the optimal contract \( (s^*, q^*) \) through calculus of variations. Let \( s^* (\vec{\theta}) = s^* (\vec{\theta}) + \alpha \cdot \varepsilon (\vec{\theta}) \) be an arbitrary perturbation of \( s^* (\cdot) \). The buyer’s best response \( m_{s,q^*} (\cdot) \) is implicitly determined by \( s (\cdot) \) through functional equation (25). Thus we need to characterize how \( m_{s,q^*} (\cdot) \) varies with respect to the perturbation of \( s^* (\cdot) \). Taking derivative with respect to \( \alpha \) at \( \alpha = 0 \) for both sides of (25) leads to

\[
\mu^{-1} \delta_b \cdot \varepsilon (\vec{\theta}) = g'' (m_{s,q^*} (\vec{\theta})) \cdot \left. \frac{dm_{s,q^*} (\vec{\theta})}{d\alpha} \right|_{\alpha=0} - g'' (\vec{p}_b) \cdot \int_{\Theta} \left. \frac{dm_{s,q^*} (\vec{\theta})}{d\alpha} \right|_{\alpha=0} dP (\vec{\theta}).
\]

Take integral of both sides and manipulate we get

\[
\int_{\Theta} \left. \frac{dm_{s,q^*} (\vec{\theta})}{d\alpha} \right|_{\alpha=0} dP (\vec{\theta}) = \mu^{-1} \delta_b \left[ 1 - \int_{\Theta} g'' (m_{s,q^*} (\vec{\theta})) \right]^{-1} dP (\vec{\theta}) \cdot g'' (\vec{p}_b) \left[ g'' (m_{s,q^*} (\vec{\theta})) \right]^{-1} \left. \varepsilon (\vec{\theta}) \right|_{\alpha=0} dP (\vec{\theta}).
\]

Combining the above two equations leads to

\[
\left. \frac{dm_{s,q^*} (\vec{\theta})}{d\alpha} \right|_{\alpha=0} = \mu^{-1} \delta_b \cdot \left[ g'' (m_{s,q^*} (\vec{\theta})) \right]^{-1} \varepsilon (\vec{\theta}) + \mu^{-1} \delta_b \int_{\Theta} \left[ g'' (m_{s,q^*} (\vec{\theta})) \right]^{-1} \varepsilon (\vec{\theta}) dP (\vec{\theta}) \cdot \left[ g'' (\vec{p}_b) \right]^{-1} \int_{\Theta} \left[ g'' (m_{s,q^*} (\vec{\theta})) \right]^{-1} dP (\vec{\theta}).
\]

This equation characterizes how the buyer’s strategy \( m_{s,q^*} (\cdot) \) responds to the perturbation of security \( s^* (\cdot) \).
Now we can calculate the variation of the seller’s expected utility $W(s, q^*)$. Taking derivative with respect to $\alpha$ at $\alpha = 0$ for both sides of (10) leads to

$$(28) \quad \frac{dW(s, q^*)}{d\alpha} \bigg|_{\alpha=0} = \int_{\Theta} \left[ q^* - \delta_s \cdot s^* \left( \tilde{\theta} \right) \right] dP \left( \tilde{\theta} \right) - \delta_s \int_{\Theta} m_{s^*, q^*} \left( \tilde{\theta} \right) P \left( \tilde{\theta} \right) dP \left( \tilde{\theta} \right).$$

Substitute (27) into (28) and manipulate we get

$$(29) \quad \frac{dW(s, q^*)}{d\alpha} \bigg|_{\alpha=0} = \int_{\Theta} r(\tilde{\theta}) : \varepsilon(\tilde{\theta}) dP \left( \tilde{\theta} \right),$$

where

$$r(\tilde{\theta}) = -\delta_s m_{s^*, q^*} \left( \tilde{\theta} \right) + \mu^{-1} \delta_h \left[ g'' \left( m_{s^*, q^*} \left( \tilde{\theta} \right) \right) \right]^{-1} \left( q^* - \delta_s \cdot s^* \left( \tilde{\theta} \right) + w \right)$$

and

$$w = \int_{\Theta} \left[ q^* - \delta_s \cdot s^* \left( \tilde{\theta} \right) \right] \left[ g'' \left( m_{s^*, q^*} \left( \tilde{\theta} \right) \right) \right]^{-1} dP \left( \tilde{\theta} \right).$$

$w$ is a constant that does not depend on $\tilde{\theta}$. Its value is endogenously determined in the equilibrium.

Let

$$A_0 = \left\{ \tilde{\theta} \in \Theta : \tilde{\theta} \neq \bar{\theta}, s^* \left( \tilde{\theta} \right) = 0 \right\},$$

$$A_1 = \left\{ \tilde{\theta} \in \Theta : \tilde{\theta} \neq \bar{\theta}, s^* \left( \tilde{\theta} \right) = 0, \sum_{n=1}^{N} \theta_n \right\},$$

and

$$A_2 = \left\{ \tilde{\theta} \in \Theta : \tilde{\theta} \neq \bar{\theta}, s^* \left( \tilde{\theta} \right) = \sum_{n=1}^{N} \theta_n \right\}.$$

Then $\{ A_0, A_1, A_2 \}$ is a partition of $\Theta \setminus \left\{ \bar{\theta} \right\}$. Since $s^* (\cdot)$ is the optimal security,

$$\frac{dW(s, q^*)}{d\alpha} \bigg|_{\alpha=0} \leq 0$$

holds for any feasible perturbation $\varepsilon(\tilde{\theta})$. Here $\varepsilon$ is feasible with respect to $s^*$ if $\exists \alpha > 0,$
s.t. $\forall \vec{\theta} \in \Theta$, $s^* \left( \vec{\theta} \right) + \alpha \cdot e \left( \vec{\theta} \right) \in \left[ 0, \sum_{n=1}^{N} \theta_n \right]$. Hence (29) implies

$$
(30) \quad r \left( \vec{\theta} \right) \begin{cases} 
\leq 0 & \text{if } \vec{\theta} \in A_0 \\
= 0 & \text{if } \vec{\theta} \in A_1 \\
\geq 0 & \text{if } \vec{\theta} \in A_2
\end{cases}
$$

Since $g$ is strictly convex, $g'' > 0$ and (30) can be rewritten as

$$
(31) \quad r \left( \vec{\theta} \right) \cdot g'' \left( m_{s^*,q^*} \left( \vec{\theta} \right) \right) = -\delta_s m_{s^*,q^*} \left( \vec{\theta} \right) g'' \left( m_{s^*,q^*} \left( \vec{\theta} \right) \right) + \mu^{-1} \delta_b \left( q^* - \delta_s \cdot s^* \left( \vec{\theta} \right) + w \right) \begin{cases} 
\leq 0 & \text{if } \vec{\theta} \in A_0 \\
= 0 & \text{if } \vec{\theta} \in A_1 \\
\geq 0 & \text{if } \vec{\theta} \in A_2
\end{cases}
$$

Recall that given the optimal contract $(s^* (\cdot), q^*)$, the buyer’s best response $m_{s^*,q^*} \left( \vec{\theta} \right)$ is characterized by

$$
(32) \quad \delta_b \cdot s^* \left( \vec{\theta} \right) - q^* = \mu \cdot \left[ g' \left( m_{s^*,q^*} \left( \vec{\theta} \right) \right) - g' \left( \mu \cdot \left( \vec{\theta} \right) \right) \right],
$$

where

$$
\mu = \int_{\Theta} m_{s^*,q^*} \left( \vec{\theta} \right) dP \left( \vec{\theta} \right)
$$

is the buyer’s unconditional probability of accepting the optimal contract $(s^* (\cdot), q^*)$.

(31)\textsuperscript{12} together with (32) determines the optimal contract $(s^* (\cdot), q^*)$. Let $m = f_1 (s)$ and $m = f_2 (s)$ be the two continuous functions implicitly defined by

$$
-\delta_s \cdot m \cdot g'' \left( m \right) + \mu^{-1} \delta_b \left( q^* - \delta_s \cdot s + w \right) = 0
$$

and

$$
\delta_b \cdot s - q^* = \mu \cdot \left[ g' \left( m \right) - g' \left( \mu \cdot \left( \vec{\theta} \right) \right) \right],
$$

respectively. $f'_1 (s) < 0$ and $f'_2 (s) > 0$ since $\left[ m \cdot g'' \left( m \right) \right]' > 0$ and $g'' \left( m \right) > 0$. By definition,

$$
m_{s^*,q^*} \left( \vec{\theta} \right) = f_1 \left( s^* \left( \vec{\theta} \right) \right) \quad \text{implies } r \left( \vec{\theta} \right) = 0.
$$

Also note that $m_{s^*,q^*} \left( \vec{\theta} \right) = f_2 \left( s^* \left( \vec{\theta} \right) \right)$ for all $\vec{\theta} \in \Theta$.

\textsuperscript{12}One may criticize that Equation (31) is just the first order condition of the seller’s optimization problem. It only characterizes the critical points. In principle, we should characterize the largest critical point. However, our argument works for any critical point and thus our results are immune to this critic.
PROPOSITION 5: \( Pr(A_0) = 0 \), where \( A_0 = \{ \theta \in \Theta : \theta \neq \theta^*, s^*(\theta) = 0 \} \).

PROOF: We first prove \( f_1(0) > f_2(0) \). If not, \( f_1(s) < f_2(s) \) for all \( s > 0 \). Thus \( \forall \theta \neq \theta^* \),

\[
\begin{align*}
r(\theta) \cdot g''(m_{s^*,q^*}(\theta)) &= -\delta_s m_{s^*,q^*}(\theta) g''(m_{s^*,q^*}(\theta)) + \mu^{-1} \delta_b (q^* - \delta_s \cdot s^*(\theta) + w) \\
&= -\delta_s \cdot f_2(s^*(\theta)) g''(f_2(s^*(\theta))) + \mu^{-1} \delta_b (q^* - \delta_s \cdot s^*(\theta) + w) \\
&< -\delta_s \cdot f_1(s^*(\theta)) g''(f_1(s^*(\theta))) + \mu^{-1} \delta_b (q^* - \delta_s \cdot s^*(\theta) + w) \\
&= 0 ,
\end{align*}
\]

where the inequality holds since \( [m \cdot g''(m)]' > 0 \). Then (31) implies \( s^*(\theta) = 0 \) almost surely. Therefore, there is no trade, which contradicts Proposition 2.

Now we know \( f_1(0) > f_2(0) \). \( \forall \theta \in A_0 \),

\[
\begin{align*}
r(\theta) \cdot g''(m_{s^*,q^*}(\theta)) &= -\delta_s m_{s^*,q^*}(\theta) g''(m_{s^*,q^*}(\theta)) + \mu^{-1} \delta_b (q^* - \delta_s \cdot s^*(\theta) + w) \\
&= -\delta_s \cdot f_2(0) g''(f_2(0)) + \mu^{-1} \delta_b (q^* - \delta_s \cdot 0 + w) \\
&> -\delta_s \cdot f_1(0) g''(f_1(0)) + \mu^{-1} \delta_b (q^* - \delta_s \cdot 0 + w) \\
&= 0 ,
\end{align*}
\]

where the second equality follows the definition that \( s^*(\theta) = 0 \) for \( \theta \in A_0 \), the last equality comes from the definition of \( f_1(s) \), and the inequality holds since \( [m \cdot g''(m)]' > 0 \). This result contradicts (31), which states \( r(\theta) \cdot g''(m_{s^*,q^*}(\theta)) \leq 0 \) for \( \theta \in A_0 \). This concludes the proof.

This proposition states that constraint \( s(\theta) \geq 0 \) never binds. As its implication, it is not optimal to issue equity residual/call option to raise liquidity. Intuitively, if the limited liability constraint

\[
s(\theta) \leq \sum_{n=1}^{N} \theta_n
\]

never binds either, both \( m_{s^*,q^*}(\theta) = f_1(s^*(\theta)) \) and \( m_{s^*,q^*}(\theta) = f_2(s^*(\theta)) \) hold for all \( \theta \). Since \( f_1'(s) < 0 \) and \( f_2'(s) > 0 \), \( f_1(s) \) and \( f_2(s) \) intersect at most once. Hence \( s^*(\theta) \) should be a constant and the buyer has no incentive to acquire information. This coincides with our intuition. If the limited liability constraint never binds, the seller would issue a security with constant repayment to avoid the buyer’s...
information acquisition. However, when the underlying cash flows are too low to support such constant, \( s^\ast (\underline{\vartheta}) \) reaches the limited liability boundary and equals \( \sum_{n=1}^{N} \theta_n \). The next proposition shows that the optimal security must be a securitized debt.

**PROPOSITION 6**: If the seller’s optimal contract induces the buyer to acquire information, it must be a securitized debt \( s^\ast (\underline{\vartheta}) = \min \left( \sum_{n=1}^{N} \theta_n, D^\ast \right) \).

**PROOF**: Let \( (\overline{s}, \overline{m}) \) be the unique intersection of \( f_1 \) and \( f_2 \) for all \( \vartheta \) such that \( \sum_{n=1}^{N} \theta_n < \overline{s} \),

\[
m_{s^\ast,q^\ast}(\vartheta) = f_2 \left( s^\ast (\vartheta) \right) < f_2 (\overline{s}) = f_1 (\overline{s}) < f_1 \left( s^\ast (\vartheta) \right) .
\]

Then

\[
\begin{align*}
& r \left( \vartheta \right) \cdot g'' \left( m_{s^\ast,q^\ast}(\vartheta) \right) \\
& = -\delta_s m_{s^\ast,q^\ast}(\vartheta) g'' \left( m_{s^\ast,q^\ast}(\vartheta) \right) + \mu^{-1} \delta_b \left( q^\ast - \delta_s \cdot s^\ast (\vartheta) + w \right) \\
& > -\delta_s \cdot f_1 \left( s^\ast (\vartheta) \right) \cdot g'' \left( f_1 \left( s^\ast (\vartheta) \right) \right) + \mu^{-1} \delta_b \left( q^\ast - \delta_s \cdot s^\ast (\vartheta) + w \right) \\
& = 0,
\end{align*}
\]

where the inequality holds since \( [m \cdot g''(m)]' > 0 \). According to (31), \( s^\ast (\vartheta) = \sum_{n=1}^{N} \theta_n \) for all \( \vartheta \) such that \( \sum_{n=1}^{N} \theta_n < \overline{s} \).

For any \( \vartheta \) such that \( \sum_{n=1}^{N} \theta_n > \overline{s} \), if \( s^\ast (\vartheta) = \sum_{n=1}^{N} \theta_n \), then (31) implies

\[
0 \leq r \left( \vartheta \right) \cdot g'' \left( m_{s^\ast,q^\ast}(\vartheta) \right) \\
= -\delta_s m_{s^\ast,q^\ast}(\vartheta) g'' \left( m_{s^\ast,q^\ast}(\vartheta) \right) + \mu^{-1} \delta_b \left( q^\ast - \delta_s \cdot s^\ast (\vartheta) + w \right) \\
= -\delta_s \cdot f_2 \left( s^\ast (\vartheta) \right) \cdot g'' \left( f_2 \left( s^\ast (\vartheta) \right) \right) + \mu^{-1} \delta_b \left( q^\ast - \delta_s \cdot s^\ast (\vartheta) + w \right) \\
< -\delta_s \cdot f_1 \left( s^\ast (\overline{s}) \right) \cdot g'' \left( f_1 \left( s^\ast (\overline{s}) \right) \right) + \mu^{-1} \delta_b \left( q^\ast - \delta_s \cdot s^\ast (\vartheta) + w \right) \\
< -\delta_s \cdot f_1 \left( s^\ast (\vartheta) \right) \cdot g'' \left( f_1 \left( s^\ast (\vartheta) \right) \right) + \mu^{-1} \delta_b \left( q^\ast - \delta_s \cdot s^\ast (\vartheta) + w \right) \\
= 0 ,
\]

which is a contradiction. Thus Proposition 5 implies \( s^\ast (\vartheta) = \overline{s} \) for all \( \vartheta \) such that \( \sum_{n=1}^{N} \theta_n > \overline{s} \).

For any \( \vartheta \) such that \( \sum_{n=1}^{N} \theta_n = \overline{s} \), \( s^\ast (\vartheta) = \overline{s} \) is a direct implication of Proposition 5.

Therefore, the optimal security is a securitized debt with face value \( \overline{s} \), i.e., \( s^\ast (\vartheta) = \min \left( \sum_{n=1}^{N} \theta_n, \overline{s} \right) \).
It is also possible that \( \tilde{s} = \infty \), i.e., \( f_1(s) \) and \( f_2(s) \) never intersects. Then the optimal security

\[
s^* (\tilde{\theta}) = \min \left( \sum_{n=1}^{N} \theta_n, \infty \right) = \sum_{n=1}^{N} \theta_n
\]

is a special securitized debt, i.e., equity. This concludes the proof.

Together with Proposition 2 and 4, this proposition enables us to conclude that pooling the assets and issuing a senior tranche is the uniquely optimal way to raise liquidity. Pooling is directly derived from the seller’s desire to maximize liquidity. It has nothing to do with the consideration of risk diversification since both agents are risk-neutral. The flat component of the optimal security results from the seller’s effort to minimize her opponent’s information acquisition. In contrast to the non-uniqueness result in (Dang, Gorton and Holmstrom 2011), we can show the unique optimality of debt because of our flexible information acquisition framework. Moreover, this flexibility also enables us to show the optimality of pooling and tranching in a broader class of environments than (Dang, Gorton and Holmstrom 2011) and without assuming a sufficiently large number of underlying assets as in (DeMarzo 2005)\(^{13}\).

In addition, our qualitative result does not rely on the distributional details of underlying assets, while most models in literature are built upon specific assumptions about the cash flows. Since the stochastic interdependence among the underlying assets could be complex and violate such assumptions, our model provides a better explanation for the prevalence of securitization in financial markets.

The security design literature usually assumes Monotone Likelihood Ratio Property (MLRP) or similar conditions to guarantee a meaningful result. Our framework justifies this assumption through endogenizing the information structure. According to Proposition 6, the optimal security \( s^* (\tilde{\theta}) \) is non-decreasing in the sum of cash flows. Proposition 1 implies that the best information structure \( m_{s^*,q^*} (\tilde{\theta}) \) is increasing in the payoff gain \( \delta_b \cdot s^* (\tilde{\theta}) - q^* \). Thus \( m_{s^*,q^*} (\tilde{\theta}) \) is also non-decreasing in the sum of the cash flows. Therefore, the larger the cash flows, the higher the probability that the buyer gets a signal asking her to accept. This can be interpreted as a generalized MLRP for multi-dimensional states.

To facilitate the analysis, the security design literature usually restrict their attention to the set of "regular" securities, which are non-decreasing in the underlying cash flows (e.g. (DeMarzo and Duffie 1999), (DeMarzo 2005)). We do not have such restriction, but show that the optimal security naturally turns out to be non-decreasing.

(Dang, Gorton and Holmstrom 2011) get debt contract uniquely optimal when their fixed information cost is zero. This can be viewed as a special case of our model where marginal cost of information acquisition vanishes.

\(^{13}\)(DeMarzo 2005) shows that the benefit of pooling achieves a theoretical maximum as the number of underlying assets approaches infinity.
C. Allocation of Bargaining Power

One may wonder if our results are sensitive to the allocation of bargaining power. The answer is no. This subsection introduces the basic setup of the case where the buyer owns the bargaining power and then presents the main results. Due to the similarity between the two cases, we omit most proofs here.

Suppose the buyer proposes the contract \( s(q) \) and the seller acquires information. Write \( m_{s,q} \) for the seller’s optimal strategy. The uninformed buyer thus enjoys expected payoff

\[
W(s,q) = \int m_{s,q}(\theta) \cdot \left[ \delta \cdot s(\theta) - q \right] dP(\theta).
\]

The buyer’s problem is to choose a feasible contract \( s(q) \) satisfying

\[
\delta_0 \leq s(q) \leq \frac{\delta_0}{\beta} \quad \text{to maximize} \quad W(s,q).
\]

Let \( (s^*(\cdot), q^*) \) denote the optimal contract for the buyer and

\[
\bar{p}_{s^*,q^*} = \int_{\Theta} m_{s^*,q^*}(\theta) dP(\theta)
\]

be the corresponding probability of trade.

**PROPOSITION 7:** \( \bar{p}_{s^*,q^*} > 0 \), i.e., trade happens with positive probability.

**PROOF:** Let \( \beta \in (\delta_0, \delta_0^{-1}, 1) \) and

\[
f(q) = \delta \cdot \mathbb{E} \min \left\{ \sum_{n=1}^{N} \theta_n, \beta \delta_0^{-1} q \right\},
\]

where the expectation is taken according to common prior \( P \). Since \( P \) is a continuous distribution and \( \beta^{-1} \delta_s \delta_b^{-1} < 1 \), there exists \( q_0 > 0 \) s.t.

\[
\Pr \left( \sum_{n=1}^{N} \theta_n \geq \beta \delta_0^{-1} q \right) > \beta^{-1} \delta_s \delta_b^{-1}
\]

for all \( q \in [0, q_0] \). Thus for any \( q \in (0, q_0) \),

\[
f'(q) = \beta \delta_s \delta_b^{-1} \int_{\{ (\theta) : \sum_{n=1}^{N} \theta_n \geq \beta \delta_0^{-1} q \}} dP(\theta)
\]

\[
= \beta \delta_s \delta_b^{-1} \cdot \Pr \left( \sum_{n=1}^{N} \theta_n \geq \beta \delta_0^{-1} q \right)
\]

\[
> \beta \delta_s \delta_b^{-1} \cdot \beta^{-1} \delta_s \delta_b^{-1} = 1.
\]

Note that

\[
f(0) = 0.
\]
which implies that
\[ f(q) > q \]
for all \( q \in (0, q_0) \).

Consider a securitized debt
\[
s\left(\vec{d}\right) = \min\left(\sum_{n=1}^{N} \theta_n, D \right)
\]
with face value \( D = \beta \delta_s^{-1} q \) and price \( q \in (0, q_0) \). Since the seller’s payoff gain from accepting this offer over rejecting it is
\[
q - \delta_s \cdot \beta \delta_s^{-1} q
\]
for all \( \vec{d} \in \Theta \), the seller will accept this offer without acquiring any information. Thus the buyer’s expected payoff from proposing \((s(\cdot), q)\) is
\[
W(s, q) = \delta_b \cdot \mathbb{E} \min\left(\sum_{n=1}^{N} \theta_n, \beta \delta_s^{-1} q \right) - q
\]
\[
= f(q) - q
\]
\[
> 0.
\]
By definition, the seller’s expected payoff through the optimal contract is \( W(s^*, q^*) \geq W(s, q) > 0 \). This directly implies \( \overline{W}_{s^*, q^*} > 0 \) since \( \overline{W}_{s^*, q^*} = 0 \) always generates zero expected payoff to the buyer.

**Proposition 8:** If the buyer’s optimal contract induces the seller to always accept it without acquiring information, it must be a securitized debt
\[
s^*\left(\vec{d}\right) = \min\left(\sum_{n=1}^{N} \theta_n, D^* \right)
\]
with price \( q^* \), where
\[
D^* = \mu \delta_s^{-1} \cdot \left[\ln \delta_b - \ln \delta_s\right] + \delta_s^{-1} q^*.
\]
q* is the unique fixed point of
\[
h(q) = \mu \ln \mathbb{E} \exp \left( \mu^{-1} \delta_s \cdot \min \left( \sum_{n=1}^{N} \theta_n, \mu \delta_s^{-1} \cdot [\ln \delta_b - \ln \delta_s] + \delta_s^{-1} q \right) \right)
\]
and the expectation is taken according to common prior P.

PROOF: The proof is very similar to that of Proposition 4 and is omitted here.

PROPOSITION 9: If the buyer’s optimal contract induces the seller to acquire information, it must be a securitized debt \( s^* \left( \bar{D} \right) = \min \left( \sum_{n=1}^{N} \theta_n, D^* \right) \).

PROOF: The proof is very similar to that of Proposition 6 and is omitted here.

Proposition 4, 6, 8 and 9 show that the optimal security is always a securitized debt, no matter who owns bargaining power. Note that the allocation of bargaining power does affect the face value and price of the debt, and thus affects the agents’ expected payoffs.

III. Discussions and Conclusions

This paper studies liquidity provision in presence of endogenous and flexible information acquisition. It shows that issuing securitized debt is always the optimal way to raising liquidity, regardless of the stochastic interdependence among the underlying assets and the allocation of bargaining power. Compared to the security design literature, our results are extremely simple and clear. We do not have to restrict our attention to the set of non-decreasing securities or assume information structures satisfying MLRP. Instead, we justify them in equilibrium.

An interesting question concerns agents’ utility functions. What happens if they are risk-averse? We do not have a clear answer at this point. The securitized debt is no longer optimal in presence of risk-aversion, since now the issuer splits the cash flows to facilitate risk sharing. In addition, issuing securitized debt is not optimal when agents have heterogeneous priors. The optimal security in this case depends on the specific form of heterogeneity.

REFERENCES


Mathematical Appendix

Proof of Proposition 1.

PROOF: Suppose $m$ is an optimal strategy. Let $\epsilon$ be any feasible perturbation function.
The payoff from the perturbed strategy $m + \alpha \cdot \varepsilon$ is

$$
V^* (m + \alpha \cdot \varepsilon) = \int_{\Theta} (m (\theta) + \alpha \cdot \varepsilon (\theta)) \cdot \Delta u (\theta) d P (\theta)
$$

$$
- \mu \cdot \left[ \int_{\Theta} g (m (\theta) + \alpha \cdot \varepsilon (\theta)) d P (\theta) - g \left( \int_{\Theta} [m (\theta) + \alpha \cdot \varepsilon (\theta)] d P (\theta) \right) \right]
$$

where $\alpha \in \mathbb{R}$, and $\varepsilon$ is feasible with respect to $m$ if $\exists \alpha > 0$, s.t. $\forall \theta \in \Theta, m (\theta) + \alpha \cdot \varepsilon (\theta) \in [0, 1]$ . Then the first order variation is

$$
\frac{dV^* (m + \alpha \cdot \varepsilon)}{d\alpha} \bigg|_{\alpha=0} = \int_{\Theta} \varepsilon (\theta) \cdot \Delta u (\theta) d P (\theta)
$$

$$
- \mu \cdot \left[ \int_{\Theta} \varepsilon (\theta) \cdot g' (m (\theta)) d P (\theta) - g' \left( \int_{\Theta} m (\theta) d P (\theta) \right) \cdot \int_{\Theta} \varepsilon (\theta) d P (\theta) \right]
$$

$$
= \int_{\Theta} \varepsilon (\theta) \cdot [\Delta u (\theta) - \mu \cdot (g' (m (\theta)) - g' (p_1))] d P (\theta)
$$

Note that

$$
\Delta u (\theta) - \mu \cdot (g' (m (\theta)) - g' (p_1))
$$

is the Frechet derivative of $V^* (\cdot)$ at $m$. Thus the tangent hyperplane at $m$ can be expressed as

$$
\left\{ \tilde{m} \in M : V^* (\tilde{m}) - V^* (m) = \int_{\Theta} \left[ \Delta u (\theta) - \mu \cdot g' (m (\theta)) + \mu \cdot g' \left( \int_{\Theta} m (\theta) d P (\theta) \right) \right] (\tilde{m} (\theta) - m (\theta)) d P (\theta) \right\}
$$

**An important observation:** since $V^* (\cdot)$ is a concave functional on $M$, $V^*$ is upper bounded by any hyperplane tangent at any $m \in M$, i.e., $\forall m, \tilde{m} \in M$,

$$
V^* (\tilde{m}) - V^* (m) \leq \int_{\Theta} \left[ \Delta u (\theta) - \mu \cdot g' (m (\theta)) + \mu \cdot g' \left( \int_{\Theta} m (\theta) d P (\theta) \right) \right] (\tilde{m} (\theta) - m (\theta)) d P (\theta)
$$

This inequality is strict when

$$
m \in M^0 \triangleq M \setminus \{ m \in M : m (\theta) \text{ is a constant almost surely} \}
$$

and $\Pr (\tilde{m} (\theta) \neq m (\theta)) > 0$, since $V^* (\cdot)$ is strictly concave on $M^0$. We will use this observation later in this proof.

The optimality of $m$ requires $\frac{dV^* (m + \alpha \cdot \varepsilon)}{d\alpha} \bigg|_{\alpha=0} \leq 0$ for all feasible perturbation $\varepsilon$. Thus
we must have

\[
\Delta u (\theta) - \mu \cdot (g' (m (\theta)) - g' (p_1)) \begin{cases} 
\geq 0 & \text{if } m (\theta) = 1 \\
= 0 & \text{if } m (\theta) \in (0, 1) \\
\leq 0 & \text{if } m (\theta) = 0
\end{cases}
\]

Note that \( \Pr (m (\theta) = 1) > 0 \) implies \( \Pr (m (\theta) = 1) = 1 \). Otherwise,

\[ p_1 = \int_\Theta m (\theta) dP (\theta) < 1 \]

and for \( \theta \in B = \{ \theta \in \Theta : m (\theta) = 1 \} \),

\[ \Delta u (\theta) - \mu \cdot (g' (m (\theta)) - g' (p_1)) = -\infty \]

since \( \lim_{x \to 1} g' (x) = \infty \). Then \( e (\theta) = -1_B \) is a feasible perturbation and

\[
\left. \frac{dV^* (m + a \cdot e)}{d\alpha} \right|_{a=0} = \int_\Theta \left[ \Delta u (\theta) - \mu \cdot (g' (m (\theta)) - g' (p_1)) \right] \cdot e (\theta) dP (\theta) = \int_B (\infty) \cdot (-1) dP (\theta) = +\infty ,
\]

which contradicts the optimality of \( m \). Thus we know that \( \Pr (m (\theta) = 1) > 0 \) if and only if \( \Pr (m (\theta) = 1) = 1 \). By the same argument, we can show that \( \Pr (m (\theta) = 0) > 0 \) if and only if \( \Pr (m (\theta) = 0) = 1 \). Therefore, the optimal strategy \( m \) must be one of the three scenarios: a) \( p_1 = 1 \), i.e., \( m (\theta) = 1 \) a.s.; b) \( p_1 = 0 \), i.e., \( m (\theta) = 0 \) a.s.; c) \( p_1 \in (0, 1) \) and \( m (\theta) \in (0, 1) \) a.s.

We first search for the sufficient condition for scenario c). According to (A1),

\[
\Delta u (\theta) - \mu \cdot (g' (m (\theta)) - g' (p_1)) = 0 \text{ a.s.}
\]

By definition,

\[ g' (x) = \ln \frac{x}{1-x} , \]

thus (A2) implies

\[ m (\theta) = \frac{p_1}{p_1 + (1 - p_1) \cdot \exp \left( -\mu^{-1} \Delta u (\theta) \right)} . \]
Let
\[
M_1 = \left\{ m(\theta, p) = \frac{p}{p + (1 - p) \cdot \exp\left(-\mu^{-1} \Delta u(\theta)\right)} : p \in [0, 1] \right\}
\]
and
\[
J(p) = \int_\Theta m(\theta, p) \, dP(\theta),
\]
then there exists \( p_1 \in [0, 1] \) such that \( m(\cdot, p_1) \in M_1 \subseteq M \) is an optimal strategy. Note that \( J(p_1) = p_1 \) is a necessary condition.

Since \( m(\cdot, p_1) \in M_1 \subseteq M \), the original problem is reduced to
\[
\max_{p \in [0, 1]} V^*(m(\cdot, p)) = \int_\Theta \Delta u(\theta) \cdot m(\theta, p) \, dP(\theta) - c(m(\cdot, p)).
\]
The first order derivative with respect to \( p \) is
\[
\frac{dV^*(m(\cdot, p))}{dp} = \int_\Theta \Delta u(\theta) \cdot \frac{\partial m(\theta, p)}{\partial p} \, dP(\theta)
\]
\[
-\mu \left[ \int_\Theta g'(m(\theta, p)) \frac{\partial m(\theta, p)}{\partial p} \, dP(\theta) - g'\left(\int_\Theta m(\theta, p) \, dP(\theta)\right) \int_\Theta \frac{\partial m(\theta, p)}{\partial p} \, dP(\theta) \right]
\]
\[
= \int_\Theta \left[ \Delta u(\theta) - \mu \cdot g'(m(\theta, p)) + \mu \cdot g'(J(p)) \right] \cdot \frac{\partial m(\theta, p)}{\partial p} \, dP(\theta).
\]
By definition,
\[
\Delta u(\theta) - \mu \cdot g'(m(\theta, p)) = -\mu \cdot g'(p),
\]
thus
\[
\frac{dV^*(m(\cdot, p))}{dp} = \int_\Theta \left[ -\mu \cdot g'(p) + \mu \cdot g'(J(p)) \right] \cdot \frac{\partial m(\theta, p)}{\partial p} \, dP(\theta)
\]
\[
= \mu \cdot \left[ g'(J(p)) - g'(p) \right] \cdot \int_\Theta \frac{\partial m(\theta, p)}{\partial p} \, dP(\theta).
\]
(A4)

Since
\[
\frac{\partial m(\theta, p)}{\partial p} = \left[ p_1 \cdot \exp\left(\frac{1}{2} \mu^{-1} \Delta u(\theta)\right) + (1 - p_1) \cdot \exp\left(-\frac{1}{2} \mu^{-1} \Delta u(\theta)\right) \right]^{-2}
\]
\[
> 0
\]
for all $\theta \in \Theta$, 
\[ \frac{dV^* (m (\cdot, p))}{dp} \geq 0 \]
if and only if
\[ g' (J (p)) - g' (p) \geq 0 . \]
Since $g'$ is strictly increasing in its argument, we have
\[ \frac{dV^* (m (\cdot, p))}{dp} \geq 0 \]
if and only if
\[ J (p) \geq p . \]
In order to be a global maximum, $m (\cdot, p_1)$ must first be a local maximum within $M_1$. This requires
\[ (A5) \quad J (p_1) = p_1 . \]
But (A5) is not sufficient. The sufficient condition for $m (\cdot, p_1)$ to be a local maximum within $M_1$ is
\[ \exists \text{ neighborhood } (p_1 - \beta, p_1 + \beta), \]
\[ \text{s.t. } J (p) \geq p \text{ for all } p \in (p_1 - \beta, p_1] \]
and $J (p) \leq p$ for all $p \in [p_1, p_1 + \beta)$. 
Note that
\[ J (0) = 0, J (1) = 1 , \]
\[ \left. \frac{dJ}{dp} \right|_{p=0} = \int_{\Theta} \exp (-1 \Delta u (\theta)) dP (\theta) \]
and
\[ \left. \frac{dJ}{dp} \right|_{p=1} = \int_{\Theta} \exp (-\mu^{-1} \Delta u (\theta)) dP (\theta) . \]
Case i): 
\[ \int_{\Theta} \exp (-1 \Delta u (\theta)) dP (\theta) > 1 \]
and
\[ \int_{\Theta} \exp (-\mu^{-1} \Delta u (\theta)) dP (\theta) > 1 . \]
In this case, $J (p) > p$ for $p$ close enough to 0 and $J (p) < p$ for $p$ close enough to 1. Since $J (p)$ is continuous, \{ $p \in (0, 1) : J (p) = p$ \} is non-empty. For any $p_1 \in
\( \{ p \in (0, 1) : J(p) = p \} \), the Frechet derivative at \( m(\cdot, p_1) \) is

\[
\begin{align*}
\Delta u(\theta) - \mu \cdot g'(m(\theta, p_1)) + \mu \cdot g'(J(p_1)) \\
= \Delta u(\theta) - \mu \cdot g'(m(\theta, p_1)) + \mu \cdot g'(p_1)
\end{align*}
\]

and thus \( m(\cdot, p_1) \) is a critical point of functional \( V^*(\cdot) \). Since \( m(\cdot, p_1) \in M^0 \), the observation mentioned above implies

\[
V^*(m) - V^*(m(\cdot, p_1))
\]

\[
< \int_\Theta \left( \Delta u(\theta) - \mu \cdot g'(m(\cdot, p_1)) + \mu \cdot g' \left( \int_\Theta m(\cdot, p_1) dP(\theta) \right) \right) (m(\theta) - m(\cdot, p_1)) dP(\theta)
\]

\[
= \int_\Theta \left( \Delta u(\theta) - \mu \cdot g'(m(\cdot, p_1)) + \mu \cdot g'(J(p_1)) \right) (m(\theta) - m(\cdot, p_1)) dP(\theta)
\]

\[
= 0
\]

for all \( m \in M \) such that \( \Pr(m(\theta) \neq m(\cdot, p_1)) > 0 \). Hence, \( V^*(m(\cdot, p_1)) \) is strictly higher than the values achieved at any other \( m \in M \), i.e., \( \{ p \in (0, 1) : J(p) = p \} = \{ p_1 \} \) and \( m(\cdot, p_1) \) is the unique global maximum.

**Case ii):**

(A6) \[ \int_\Theta \exp(\mu^{-1} \Delta u(\theta)) dP(\theta) > 1 \]

and

(A7) \[ \int_\Theta \exp(-\mu^{-1} \Delta u(\theta)) dP(\theta) \leq 1 . \]

(A6) implies \( J(p) > p \) for \( p \) close enough to 0. Note that

\[
\left. \frac{d^2 J}{dp^2} \right|_{p=1} = -2 \cdot \int_\Theta \left[ \exp(-\mu^{-1} \Delta u(\theta)) - \exp(-2\mu^{-1} \Delta u(\theta)) \right] dP(\theta)
\]

\[
= -2 \cdot \left[ \mathbb{E} \exp(-\mu^{-1} \Delta u(\theta)) - \mathbb{E} \exp(-2\mu^{-1} \Delta u(\theta)) \right] ,
\]

where the expectation is taken according to prior \( P \). Since

\[ f(x) = x^2 \]

is a strictly convex function, Jensen’s inequality implies

\[ \mathbb{E} \exp(-\mu^{-1} \Delta u(\theta)) \geq \mathbb{E} \exp(-2\mu^{-1} \Delta u(\theta)) . \]

The inequality is not strict only if \( \Delta u(\theta) = \text{constant} \) almost surely. Since \( \mathbb{E} \exp(-\mu^{-1} \Delta u(\theta)) \leq \)}
1, this constant must be non-negative. Moreover, since \( \Pr (\Delta u (\theta) \neq 0) > 0 \), this constant must be strictly positive. Thus

\[
\mathbb{E} \exp (-\mu^{-1} \Delta u (\theta)) > \mathbb{E} \exp (-2\mu^{-1} \Delta u (\theta))
\]

and

\[
(A8) \quad \frac{d^2 J}{dp^2} \bigg|_{p=1} < 0.
\]

Together with (A7), (A8) implies \( J(p) > p \) for \( p \) close enough to 1. Thus there exists \( \epsilon > 0 \), s.t. \( J(p) > p \) for all \( p \in [0, \epsilon] \cup [1 - \epsilon, 1] \).

We claim that \( J(p) > p \) for all \( p \in (0, 1) \). If this is not true, let \( p_1 = \sup \{ p \in (0, 1) : J(p) \leq p \} \). The continuity of \( J(p) \) implies \( J(p_1) = p_1 \). Thus \( m(\cdot, p_1) \in M^a \) and it is a critical point of functional \( V^*(\cdot) \). By the same argument as in Case i), \( m(\cdot, p_1) \) is the unique global maximum. However, by definition, \( p_1 < 1 - \epsilon \) and \( J(p) > p \) for all \( p \in (p_1, 1) \). Then \( V^*(m(\cdot, p)) > V^*(m(\cdot, p_1)) \) for all \( p \in (p_1, 1) \) since \( \frac{dV^*(m(\cdot, p))}{dp} \) is of the same sign as \( J(p) - p \). This contradicts the unique optimality of \( m(\cdot, p_1) \). Therefore, \( J(p) > p \) for all \( p \in (0, 1) \) and the optimal strategy cannot be an interior point of \( M \) (i.e., it cannot be the case \( p_1 \in (0, 1) \)). Then according to our previous discussion, only scenario a) that \( p_1 = 1 \) and scenario b) that \( p_1 = 0 \) are possible. Since we have shown \( J(p) > p \) for all \( p \in (0, 1) \), we know that

\[
V^*(m(\cdot, 1)) > V^*(m(\cdot, 0)).
\]

Hence, \( p_1 = 1 \), i.e., \( m(\theta) = 1 \) a.s. is the unique optimal strategy.

\textbf{case iii):}

\[
\int_{\Theta} \exp \left( \mu^{-1} \Delta u (\theta) \right) dP (\theta) \leq 1
\]

and

\[
\int_{\Theta} \exp \left( -\mu^{-1} \Delta u (\theta) \right) dP (\theta) > 1.
\]

In this case, by the same argument as in case ii), \( m(\theta) = 0 \) a.s. is the unique optimal strategy.

Now we show that it is impossible to have the case

\[
(A9) \quad \int_{\Theta} \exp \left( \mu^{-1} \Delta u (\theta) \right) dP (\theta) \leq 1
\]

and

\[
(A10) \quad \int_{\Theta} \exp \left( -\mu^{-1} \Delta u (\theta) \right) dP (\theta) \leq 1.
\]
Since
\[ f(x) = x^{-1} \]
is strictly convex for \( x > 0 \), Jensen’s inequality implies
\[
\int_{\Theta} \exp \left(-\mu^{-1} \Delta u(\theta)\right) dP(\theta) \geq \left[ \int_{\Theta} \exp \left(\mu^{-1} \Delta u(\theta)\right) dP(\theta) \right]^{-1}.
\]
The inequality is not strict only if \( \Delta u(\theta) = \text{constant almost surely} \). If this is true, (A9) and (A10) implies \( \Delta u(\theta) = 0 \) almost surely. This is the trivial case excluded by our assumption. Thus
\[
\int_{\Theta} \exp \left(-\mu^{-1} \Delta u(\theta)\right) dP(\theta) > \left[ \int_{\Theta} \exp \left(\mu^{-1} \Delta u(\theta)\right) dP(\theta) \right]^{-1}
\]
and (A9) and (A10) cannot be simultaneously satisfied.

Since case i), ii) and iii) exhaust all possibilities, for each case, the corresponding conditions are not only sufficient but also necessary.

The uniqueness of the optimal strategy is proved in each case.

This concludes the proof.