Investment and Bargaining

Adam Meirowitz
Department of Politics
040 Corwin Hall
Princeton University
Princeton, NJ 08540
phone: (609)258-4859
fax: (609)258-1110
ameirowi@princeton.edu

Kristopher W. Ramsay
Department of Politics
033 Corwin Hall
Princeton University
Princeton, NJ 08540
phone:(609)258-2960
kramsay@princeton.edu

*We thank Sandeep Baliga, Ethan Bueno de Mesquita, John Duggan, Thomas Palfrey, Rahul Pandharipande, John Patty, Charlie Plott, Tom Romer Larry Samuelson, and Bruno Strulovic for comments on earlier versions of this paper. We also thank participants at Caltech, Columbia, Essex, LSE, Nuffield, Princeton, Rutgers, Rochester, Vanderbilt and the Midwest Political Science Association Annual Conference for helpful comments. We are also quite appreciative of comments and discussions, as well as support from, the 2009 European Summer Symposium in Economic Theory at the Study Center Gerzensee. Peter Buisseret, Nikolaj Harmon, and SeHyoun Ahn provided excellent research assistance.
Abstract

We consider bargaining between two players who may invest \textit{ex ante} in their agreement and disagreement payoffs. We characterize necessary conditions on equilibrium investment strategies in this environment, describe how investments and the probability of outcomes must vary across mechanisms, and specify what equilibrium conditions imply for constraints on various aspects of the design environment. In the case of private values any two trading rules that induce the same equilibrium lotteries over valuations induce the same probability of trade. We also show that equilibrium descriptions are fragile in the sense that descriptions that cannot be supported by any equilibrium investment decisions are dense in the space of problems. We exhibit an approach to recovering cost functions that support the primitives of a Bayesian Mechanism design problem. By way of an example, we use an unraveling argument to show that uniform distributions over valuations cannot emerge from equilibria investments to the Chatterjee-Samuelson bilateral trade mechanism for any strictly increasing cost functions.
1 Introduction

Two classic examples of bargaining involve a buyer and seller negotiating the price of a productive technology (Chatterjee and Samuelson, 1983) and diplomatic negotiations between two countries over how to divide a disputed territory (Schelling, 1960). Whether the technology is sold or the dispute settled peacefully depends on many factors, including the rules of negotiations, players’ values for agreement and disagreement, and any underlying uncertainty regarding payoffs. Often in these contexts the parties can take hidden actions that will influence their payoffs from agreement, disagreement, or both. Before negotiations the buyer and seller can obtain options on inputs or complimentary technologies that increase the value to them of the item under negotiation. A seller may invest in preventative maintenance or defer such costs influencing the value of the item. Alternatively, the buyer and seller may line up alternative technologies raising their respective disagreement and agreement values. Countries can invest in arms and alliances influencing the attractiveness of war. In equilibrium the investment decisions may influence bargaining behavior, and expectations about bargaining behavior may influence investment decisions. Standard analysis of bargaining suppresses this fact and assumes instead payoffs for bargaining outcomes are exogenous, though not necessarily commonly known.¹ In this paper, we consider the implications of pre-bargaining investments in the agreement and disagreement values on the equilibria of bargaining problems allowing for equilibria in which strategic uncertainty emerges endogenously. The main objective is to characterize consistent descriptions of equilibrium investment and bargaining strategies as well as the related exogenous technology and primitives.

Why focus on extending the canonical model of bargaining to include pre-bargaining investment? First, we believe that this modeling strategy captures an empirically important and interesting feature of some real-life bargaining problems. Second, imposing equilibrium constraints on investments, and therefore the distributions of payoff values, can significantly limit which distributions over bargaining outcomes are consistent with Nash equilibrium behavior when compared to predictions of Bayesian-Nash

¹Analysis of the hold-up problem allows for investment by at least one player but in these problems the investments are typically observable or at least predictable in equilibrium. Exceptions are discussed below.
equilibria from models in which the prior beliefs over types are not pinned down. If one is agnostic about the cost function then, not surprisingly, many different lotteries over behavior are possible. Somewhat surprisingly, however, even without constraints on the cost functions, we show that descriptions of behavior that are not consistent with equilibrium play are still prevalent. More importantly, when the substance of the problem justifies certain restrictions on the exogenous cost functions (for example the idea that increasing one’s agreement value requires larger investments) the approach provides fairly strong restrictions on equilibrium behavior. So while the standard Bayesian approach to mechanism design or bargaining asks the analyst to start with a primitive description of beliefs, this approach asks the analyst to start with a primitive description of the cost function (something that might be easier to measure) and then lets equilibrium conditions build off of the assumed cost functions to generate beliefs about the valuations in the bargaining problem. In this paper, we limit ourselves to models in which the starting cost functions are known by the players, but one could imagine building off these results to introduce exogenous uncertainty about the costs.

To illustrate the point of this paper, consider the following simple example. Suppose two countries are involved in a dispute over a prize of value 1 and each country can have a strong or weak military. Typically, one would model the situation by assuming that each nation had a “type” that is either strong or weak. In our example, the countries interact in a simple way. Suppose that they have brought their dispute to a mediator, this mediator has proposal power, and the mediator proposes and enforce a settlement of 1/2 to each country; but the mediator cannot broker any other settlement. If both countries accept this settlement then they receive their share and the game ends. If any country refuses the settlement then the mediator walks away and a war over the prize occurs. The expected payoffs from war are assumed to be interdependent and hinge on the profile of types. We assume that the expected payoffs from conflict are given by the matrix in the left panel of Figure 1, where the first entry in each cell is the expected payoff of country one and the second entry is the expected payoff of country two. A natural set of assumptions is that $0 < b < s < \frac{1}{2} < w < w + l < 1$.

\footnote{This structure captures the ideas that fighting is inefficient (destroys resources), wars between}
### Table: Contest function and Mediation game form

<table>
<thead>
<tr>
<th></th>
<th>weak</th>
<th>strong</th>
<th>Mediation game form</th>
</tr>
</thead>
<tbody>
<tr>
<td>weak</td>
<td>s, s</td>
<td>l, w</td>
<td>Settle: 1/2, 1/2, war</td>
</tr>
<tr>
<td>strong</td>
<td>w, l</td>
<td>b, b</td>
<td>Fight: war, war</td>
</tr>
</tbody>
</table>

Figure 1: Example

A typical way to proceed would be to assume that the types are generated by an exogenous distribution; for example we might assume that each country is strong with probability $\pi$ and that the types are independent random variables. This game has two equilibria. In one equilibrium both types of both countries choose to fight and war occurs with probability one. In the only other Bayesian Nash equilibrium, strong types reject the mediator’s proposed settlement and weak types accept it. Here the ex ante probability of war is $1 - (1 - \pi)^2$.

Now instead suppose each country’s strength is the result of a hidden strategic choice and assume that it is costly to become strong. That is, a country must pay $c$ to have a strong military.\(^3\) In this game we replace the exogenous distribution over types with an exogenous cost and allow for the strengths to be endogenously determined. This game has two equilibria. In pure strategies, both sides arming and fighting is an equilibrium and, like before, the ex ante probability of war is one. The other equilibrium involves mixing in the investment decisions and thus results in strategic uncertainty. The usual mixing condition implies a country choses to be strong with probability $\frac{1 - 2(w - c)}{1 - 2(w - b + l)}$. It is then easy to verify that it is a Nash equilibrium for each country to mix with probabilities equal to $(\frac{1 - 2(w - c)}{1 - 2(w - b + l)}, 1 - \frac{1 - 2(w - c)}{1 - 2(w - b + l)})$, fights when strong and accepts the mediator’s proposal when weak. This is the only equilibrium in which war occurs with probability less than one. The above expression, for the probability of arming, illustrates that equilibrium play imposes a relationship between the cost function and the probability that a country is strong. One could go back and forth between checking what assumptions about $c$ are consistent with equilibrium

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\(^3\)Assume $0 < c < \min\{w - 1/2, b - l\}$.

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**strong states are more destructive than wars between weak states, and that it is better to be strong than weak if you are going to fight.**
probabilities of arming. For the probability of strong to be 1/2 we need $c = (w + b - l)/2 - 1/4$.

In this equilibrium the *ex ante* probability of war is

$$1 - \left[ 1 - \left( \frac{1 - 2(w - c)}{1 - 2(w - b + l)} \right) \right]^2,$$

which depends on the primitives, $(w, b, l)$. This is in contrast to the probability of war in the comparable equilibrium in the Bayesian game, $1 - (1 - \pi)^2$ which does not.

As a matter of counting parameters, the exchange of the prior for the cost of investing may seem an even trade, but specifying cost functions and deriving beliefs over valuations may be more natural. Measuring costs is something that economists do with varying precision quite regularly. Assessing the cost functions is a matter of understanding the technology relevant for a given problem. Measuring beliefs, one the other hand, may be more challenging or less natural. The hidden actions approach, thus, allows equilibrium analysis to inform the choice of priors over valuations when it is possible to specify the cost functions (or beliefs about cost functions in a more general setting). Finally, the selection of $c$ could be connected with the design problem in the following way. Using the equation (1) we might ask what value(s) of $c$ makes the equilibrium behavior consistent with a particular conjecture about the equilibrium probability of war or joint distribution over strength levels. That is, we may seek $c$’s that are consistent with a particular description of behavior.

Our observation that expectations about the negotiation process can influence investment decisions is not entirely surprising. The literature on the hold-up problem considers investment by buyers and or sellers prior to trade and makes a similar point (Grossman and Hart, 1986; Gul, 2001; Segal and Whinston, 2002). In the hold-up problem the investments are typically observable. Here, we consider problems where the investments influence the value of the agreement or disagreement payoffs and where actions are hidden. The paper in this literature that is closest to ours is Rogerson (1992). Rogerson considers the mechanism design problem where there is some investment influencing types, but focuses on investments in private values and is primarily concerned with showing that the d’Aspremont and Grard-Varet (1979) mechanism provides the first-best solution to the hold-up problem in the presence.
of *ex ante* investments. A key difference is that Rogerson assumes that a unique investment level is optimal and thus investment decisions cannot be the source of strategic uncertainty. In contrast we allow for the possibility that, in equilibrium, players cannot perfectly predict the investment decisions of each other. We also allow for interdependent values (investments with externalities).

Our analysis is also related to a law and economics paper by Plott (1987). In that article Plott considers a model in which two parties invest in legal fees and then a contest function determines the trial winner. He shows that the choice of legal rules, in particular who pays the legal fees after a trial, can influence the quantity of legal expenses that the parties absorb and points out that the English rule may be inefficient. The Plott model differs from ours in that it does not model interactions in the trial or allow for settlements to be reached. In this way the Plott model is a special mechanism in our set-up where the probability of disagreement is always one.

The current paper draws on progress made in a related paper in the study of bargaining and war. Meirowitz and Sartori (2008) consider models in which military investment decisions are hidden actions, players bargain after making their investment decisions, and in the case of bargaining failure there is war. Their analysis of the war problem shows the possibility of war in equilibrium is necessary and sufficient for military investments to be in mixed strategies. Similarly, Jackson and Morelli (2008) characterize mixed strategy equilibria in a war game when investment decisions are observable. Thus, in at least one interesting context it is important to capture equilibrium investment decisions that are not perfectly predictable.

Below we begin by defining the general bargaining problem and characterizing a consistent view of investment in this setting. In turning to the analysis, we start by taking as given a fixed lottery over types and analyzing the induced problem of mechanism design with interdependent values. In an approach analogous to backward induction, we then use the results from this analysis to characterize a necessary condition on equilibrium investment strategies and bargaining behavior. We then turn to some comparative statics and provide results on the possibility of finding consistent descriptions of behavior given fixed primitives, and the possibility of finding primitives to support given descriptions of behavior. We end by considering two examples, bargaining in the shadow of war and bilateral trade with private values.
2 Model

The first step is to define the set of situations for which our analysis applies. Consider the interaction between two players in anticipation of a negotiation. Each player, \( i \in \{1, 2\} \) must first select a level of investment \( a_i \in \mathbb{R}_+ \) that will contribute to their payoffs. In the case of international conflict, for example, this investment may be spent on arms and influence the chance of winning a war. In the case of bilateral trade, the seller may invest in the quality of the good or secure options on possible replacements, whereas the buyer might invest in goods that are complimentary to the item as well as possible substitutes. These investments influence the values of reaching a settlement. To allow investments to be in mixed strategies we write \( F_i(\cdot) \) to capture the distribution function of \( a_i \). We assume that the cost of investment \( a_i \) is given by \( c_i(a_i) \) where \( c_i(\cdot) \) is a strictly increasing and differentiable function. By \( c_i'(a_i) \) we denote the first derivative of the cost at \( a_i \) and by \( c_i^{-1}(\cdot) \) we denote the inverse of the cost function. The investment choices are assumed to be hidden actions—player \( i \) knows its choice of \( a_i \) but it does not observe the choice by player \( -i \). The players then negotiate over an allocation \( x \) and transfers \( t_1 \) and \( t_2 \). We limit ourselves to bargaining problems in which there are two possible types of allocations or outcomes, there is an agreement allocation and a disagreement allocation. That is, there is trade or no trade, sale or no sale, a peaceful settlement or war. To keep this clear we decompose payoffs over the allocation \( x(\cdot, \cdot) \) into two functions \( v^A_i(\cdot, \cdot) \), \( v^D_i(\cdot, \cdot) \), where \( v^A_i \) is player \( i \)'s investment contingent value of agreement and \( v^D_i \) is player \( i \)'s investment contingent value of disagreement. We assume that \( v^A_i \) and \( v^D_i \) are bounded and at least twice continuously differentiable for each player.

Given agreement with transfers, \( t_1 \) and \( t_2 \), and a pair of investments \((a_1, a_2)\), player \( i \)'s payoff is \( v^A_i(a_i, a_{-i}) + t_i - c_i(a_i) \). For disagreement the payoff is \( v^D_i(a_i, a_{-i}) + t_i - c_i(a_i) \). For example, if we were to map our notation into Chatterjee and Samuelson’s model of bilateral trade, for the seller \( v^A_i(a_s, a_b) \) is always zero giving a payoff to trade of \( t_s - c_s(a_s) \). For the buyer \( v^A_i(a_s, a_b) \) is her value of the good and the payoff to trade is \( v^A_b(a_s, a_b) + t_b - c_b(a_b) \), with the price paid by the buyer being \( t_b \leq 0 \) and the payment received by the seller being \( t_s \geq 0 \).

We proceed without assuming any particular model of negotiation. We think of a
negotiation procedure, protocol, game or “institution” as a sequence of interactions that must eventually either lead to an agreement or result in the disagreement outcome. We draw upon the revelation principle to establish results about a large class of games in which the investment decisions are hidden actions by focusing on direct revelation mechanisms in which after selecting their investment levels the players make unverifiable reports.

A direct revelation mechanism, has message spaces of \( M_1 = M_2 = \mathbb{R}_+ \) and a triple of mappings

\[
    \begin{align*}
        t_1 & : M_1 \times M_2 \rightarrow \mathbb{R} \\
        t_2 & : M_1 \times M_2 \rightarrow \mathbb{R} \\
        q & : M_1 \times M_2 \rightarrow [0, 1].
    \end{align*}
\]

So a direct revelation mechanism in our setting is a cheap talk game where \( M_i \) is player \( i \)'s possible hidden actions, \( q \) is the probability of disagreement, and \( t_i \) is \( i \)'s report contingent transfer. We also note that in our setting the allocation (agreement or disagreement) payoffs depend only on the level of investment that each player has made in the investment stage, not on the reports of the players. If we think of the problem without the revelation principle, then the assumption is that allocation payoffs do not depend on players bargaining strategies. The lottery over agreement, of course, might depend on bargaining strategies.

To simplify the exposition, we sometimes use \( c \) to denote the functions \( (c_1, c_2) \), \( t \) to denote the functions \( (t_1, t_2) \), \( v_i \) to denote the functions \( (v^D_i, v^A_i) \) and \( v \) to denote the functions \( (v_1, v_2) \). Before turning to the analysis, a few observations are worth making. We do not require that the transfers satisfy budget balance. We also do not require that transfers are non-negative. Finally, we do not require that the players are willing to accept settlements that are distributed by the mechanism. We remain agnostic about these issues. We take this approach because we are interested in the consequences of incentive compatibility and optimality of investments. Additional structure and concern about budget balance or participation constraints would impose additional restrictions and might involve conditions that are even stronger than those contained here. Importantly, however, the results we prove cannot be relaxed by
adding more structure of this form. Since we do not focus on sufficient conditions for particular types of equilibria (or even existence of equilibria) this additional structure is not needed. Subsequent work that proceeds in the other direction will surely need to deal with these added complications.

3 Implications of Incentive Compatibility

The analytic convenience of the revelation principle is that it allows us to learn about equilibria to any mechanism by focusing on just “truthful” equilibria to direct mechanisms. The latter are studied by way of incentive compatibility conditions, which ensure that players are willing to truthfully report their private information to the mediator in a direct mechanism. We begin with a fairly standard description of incentive compatible behavior in a direct mechanism, treating the distribution functions \( F_i \) as fixed. Once this aspect of the problem is pinned down we take a step back endogenizing these distributions.

Let \( F_i \) be player \( i \)'s mixed strategy equilibrium distribution over the hidden action. Recall our direct mechanism is a pair of functions \( t_i(m_i, m_j) : \mathbb{R}_+^2 \rightarrow \mathbb{R} \) that describes the report contingent transfer to \( i \) and a function \( q(m_i, m_j) : \mathbb{R}_+^2 \rightarrow [0, 1] \) that determines the probability of disagreement. Expected utility (net of investment costs) to \( i \) of making a report \( m_i \) in this direct mechanism, given investment \( a_i \), can then be written as

\[
U_i(m_i|a_i) = \int [(1 - q(m_i, m_j))v_i^A(a_i, m_j) + q(m_i, m_j)v_i^D(a_i, m_j) + t_i(m_i, m_j)]dF_j(m_j).
\]

It is convenient to define

\[
\overline{t}_i(m_i) = \int t_i(m_i, m_j)dF_j(m_j),
\]

\[
\overline{v}_i^A(m_i|a_i) = \int (1 - q(m_i, m_j))v_i^A(a_i, m_j)dF_j(m_j),
\]

\[
\overline{v}_i^D(m_i|a_i) = \int q(m_i, m_j)v_i^D(a_i, m_j)dF_j(m_j),
\]

and we can describe a player’s interim expected utility of a report \( m_i \) when his
investment is $a_i$ as

$$U_i(m_i|a_i) = \bar{I}_i(m_i) + \bar{v}_i^A(m_i|a_i) + \bar{v}_i^D(m_i|a_i)$$

In a slight abuse of notation, let $U_i(a_i) = U_i(a_i|a_i)$. While we will impose no particular structure on $q(\cdot, \cdot)$ and $F_1, F_2$, as our goal is to determine things that must be true in any equilibrium, we will require that equilibria are well behaved in the sense that expected utilities are always well-defined (Lebesgue integrable functions).

The direct revelation mechanism framework is useful as it allows us to characterize incentives across game forms and various equilibria. Throughout the paper we invoke the following revelation principle.

**Revelation Principle** If there exists a game with equilibrium investing decisions given by the mixtures $F_1$ and $F_2$ and the lottery $G(t_1, t_2, p_1, p_2)$ over transfers and allocation payoffs, then there is a direct mechanism possessing an equilibrium in which investing strategies are given by $F_1$ and $F_2$ and the states report truthfully $m_i(a_i) = a_i$, which induces the same lottery over the outcomes.

We omit the proof, as it is straightforward. For fixed investment strategies the argument involves the standard composition strategy as found in Myerson (1979). Since investment decisions are privately observed and reports are unverifiable this first stage introduces no additional complications. Without loss of generality we will proceed by looking at equilibrium incentives in direct mechanisms and focus on Bayesian Nash equilibria to the induced games.

We note a convenient feature of the supports of investment strategies. Since investments matter only because they influence the allocation payoff $v_i^A$ and $v_i^D$, and they impose a cost $c_i(\cdot)$, any investment $a_i$ with

$$c_i(a_i) > \overline{\pi}_i = \max\left\{ \sup_{a_{-i}} v_i^A(a_i, a_{-i}), \sup_{a_i, a_{-i}} v_i^D(a_i, a_{-i}) \right\}$$

is strictly dominated by $a_i' = 0$. Since $c_i(\cdot)$ is increasing in $a_i$ and each component of $v$ is bounded, a strict dominance argument implies equilibrium investments will have support contained in the interval $[0, c_i^{-1}(\overline{\pi}_i)]$. We can then conclude that equilibrium investments always have compact support.
Our first result characterizes the value of playing a bargaining game as a function of the pre-play investment choices. The proof shows that the envelope theorem of Milgrom and Segal (2002) applies to our environment with probability distributions that arise from mixed strategies and $q$ functions that need not be continuous.

**Theorem 1** Let $[a_i, \bar{a}_i]$ be the support of $F_i$. If $<t_1, t_2, q>$ is an equilibrium given these distributions then (1) for almost every $a_i$ in $[a_i, \bar{a}_i]$ the derivative of the value function exists and net of costs it is given by

\[
U'_i(a_i) = \int_{a_i}^{\bar{a}_i} \left[ \frac{\partial v^D_i(a_i, a_j)}{\partial a_i} q(a_i, a_j) + \frac{\partial v^A_i(a_i, a_j)}{\partial a_i} (1 - q(a_i, a_j)) \right] dF_j(a_j)
\]

and (2) net of costs, the value function is given by

\[
U_i(\bar{a}_i) = U_i(a_i) + \int_{a_i}^{\bar{a}_i} \int_{a_i}^{\bar{a}_i} \left[ \frac{\partial v^D_i(a_i, a_j)}{\partial a_i} q(a_i, a_j) + \frac{\partial v^A_i(a_i, a_j)}{\partial a_i} (1 - q(a_i, a_j)) \right] dF_j(a_j) da_i
\]

**Proof.** Fix $F_1, F_2$ and suppose $<t_1', t_2', q'>$ is an equilibrium to such a game. By the Revelation Principle there is a direct mechanism $<t_1, t_2, q>$ such that players truthfully report and the mechanism induces the same lottery over outcomes. We use Milgrom and Segal (2002, Thrm 2) to establish claims (1) and (2).

Milgrom and Segal state three sufficient conditions (which we rewrite in our notation):

(i) $U_i(m_i|a_i)$ is absolutely continuous and differentiable in $a_i$ for all $m_i$;

(ii) there exists an integrable function $g : [a_i, \bar{a}_i] \to \mathbb{R}$ such that $\left| \frac{\partial U_i(m_i|a_i)}{\partial a_i} \right| \leq g(a_i)$ for all $m_i$ and almost every $a_i$; and

(iii) that an optimal response, $m_i$, exists for each type $a_i$.

Condition (iii) is satisfied in any incentive compatible direct mechanism. The rest of the proof focuses on verifying that the first two conditions are satisfied and
proceeds in three steps. First we show that we can write $\frac{\partial U_i}{\partial a_i}(m_i|a_i)$ as
\[
\frac{\partial}{\partial a_i} \int [(1 - q(m_i, m_j))v^A_i(a_i, m_j) + q(m_i, m_j)v^D_i(a_i, m_j) + t_i(m_i, m_j)]dF_j(m_j),
\]
\[
= \int [(1 - q(m_i, m_j))\frac{\partial v^A_i(a_i, m_j)}{\partial a_i} + q(m_i, m_j)\frac{\partial v^D_i(a_i, m_j)}{\partial a_i}]dF_j(m_j).
\]

Second we show that $U_i(m_i|a_i)$ is Lipschitz continuous in $a_i$, which reduces to showing that the derivative of $U_i$ with respect to $a_i$ is bounded. We then use these conclusions to show that $U_i(m_i|a_i)$ is absolutely continuous in $a_i$ and bounded by a linear function and thus equal to the integral of its derivative almost everywhere.

Observe that the investment level only enters $U_i(m_i|a_i)$ through the term $v^A_i(m_i|a_i)$ and $v^D_i(m_i|a_i)$, so we can simplify things and ignore the term $\bar{t}_i(m_i)$.

**Lemma 1** $\frac{\partial U_i}{\partial a_i}(m_i|a_i) = \int [(1 - q(m_i, m_j))\frac{\partial v^A_i(a_i, m_j)}{\partial a_i} + q(m_i, m_j)\frac{\partial v^D_i(a_i, m_j)}{\partial a_i}]dF_j(m_j)$ at every $a_i, m_i$.

**Proof.** We verify that standard conditions for interchanging the order of integration and differentiation are satisfied. We use the versions presented in Durrett (1995, Thrm 9.1):

(a) $\int \left[[(1 - q(m_i, m_j))v^A_i(a_i, m_j) + q(m_i, m_j)v^D_i(a_i, m_j)]\right]dF_j(m_j) < \infty$;

(b) $(1 - q(m_i, m_j))\frac{\partial v^A_i(a_i, m_j)}{\partial a_i} + q(m_i, m_j)\frac{\partial v^D_i(a_i, m_j)}{\partial a_i}$ exists and is continuous in $a_i$ for each $m_i, m_j$;

(c) $\int [(1 - q(m_i, m_j))\frac{\partial v^A_i(a_i, m_j)}{\partial a_i} + q(m_i, m_j)\frac{\partial v^D_i(a_i, m_j)}{\partial a_i}]dF_j(m_j)$ is continuous in $a_i$ for every $m_i$;

(d) and for each $a_i, m_i$ there is some $\delta > 0$ s.t.
\[
\int \int_{-\delta}^{\delta} \left[1 - q(m_i, m_j)\right]\frac{\partial v^A_i(a_i + \theta, m_j)}{\partial a_i} + q(m_i, m_j)\frac{\partial v^D_i(a_i + \theta, m_j)}{\partial a_i} d\theta dF_j(m_j)
\]
is finite.
Condition (a) holds as \( v_i^A, v_i^D \), and \( q \) are non-negative so
\[
\int \left| \left( 1 - q(m_i, m_j) \right) v_i^A(a_i, m_j) + q(m_i, m_j) v_i^D(a_i, m_j) \right| dF_j(m_j) \\
= \int \left| (1 - q(m_i, m_j)) v_i^A(a_i, m_j) + q(m_i, m_j) v_i^D(a_i, m_j) \right| dF_j(m_j)
\]
where we have assumed that the latter is finite.

Conditions (b) and (c) follow from the assumption that \( v_i^A \) and \( v_i^D \) are continuously differentiable in their arguments and \( q \) is a probability for each \( m_i, m_j \).

Condition (d) is verified as follows. First consider \( v_i^D \). Since \( v_i^D \) is continuously differentiable in all its arguments, we know that its first derivative is continuous. Since the support of the investment strategy must be a compact set, this derivative attains a finite maximum. Thus we may concluded that the derivative of \( v_i^D \) is bounded and \( v_i^D \) is Lipschitz continuous. As \( q \) is also bounded above by 1 and the mixed strategy \( F_j \) integrates to 1, there exists a constant \( M \) s.t.
\[
\int \int_\delta - \delta \left| \frac{\partial v_i^D(a_i + \theta, m_j)}{\partial a_i} q(m_i, m_j) \right| d\theta dF_j(m_j) < \\
\int \int_\delta - \delta M \theta d\theta dF_j(m_j) = \\
\int 2\delta M dF_j(m_j) = 2\delta M < \infty,
\]
a symmetric argument holds for \( v_i^A \) and the linearity of the integral means we can satisfy the last condition of Durrett. Differentiating the function under the integral proves the identity. ■

Next we show that \( U_i(m_i | a_i) \) is also Lipschitz continuous in \( a_i \).

**Lemma 2** \( U_i(m_i | a_i) \) is Lipschitz continuous.

**Proof.** Because \( v_i^D \) is Lipschitz continuous and the arguments lie in a bounded set, there exists a number \( k \) such that
\[
\frac{\partial v_i^D}{\partial a_i}(a_i, m_j) < k.
\]
Moreover since $0 \leq q(m_i, m_j) \leq 1$, $F_j$ integrates to 1, we get
\[
\int_{m_j}^{\pi_j} \frac{\partial v_i^D(a_i, t)}{\partial a_i} q(m_i, t) dF_j(t) < k \int_{m_j}^{\pi_j} q(m_i, t) dF_j(t) < k[\pi_j - a_j].
\]
A parallel argument holds for $v_i^A$, and given the definition of $\frac{\partial U_i}{\partial m_i}$ in Lemma 1, the derivative of $U_i(m_i | a_i)$ with respect to $a_i$ is bounded and the utility is Lipschitz continuous.

So Lemma 2 implies that (ii) is satisfied with $g(a_i) = Ka_i$ for some constant $K$. Moreover, since Lipschitz continuity implies absolute continuity on the interval and Lemma 1 establishes the differentiability of $U_i(m_i | a_i)$ for each $m_i$, condition (i) is satisfied, completing the proof.

We now turn to the study of what types of investment strategies are possible in an equilibrium. We find that the equilibrium conditions from strategic investment pin down a number of characteristics of the bargaining problem.

**Theorem 2** In any equilibrium to any bargaining game, if $a_i$ is in the support of $i$’s mixed strategy then
\[
c_i'(a_i) = \int_{m_j}^{\pi_j} \left[ \frac{\partial v_i^D(a_i, a_j)}{\partial a_i} q(a_i, a_j) + \frac{\partial v_i^A(a_i, a_j)}{\partial a_i} (1 - q(a_i, a_j)) \right] dF_j(a_j)
\]  

Proof. Suppose $\langle t'_1, t'_2, q', F_1, F_2 \rangle$ is an equilibrium. By the Revelation Principle there is a direct mechanism $(t, q)$ such that players truthfully report and the mechanism induces the same lottery over outcomes. We focus on such a direct mechanism. For $F_1, F_2$ to constitute equilibrium mixed strategies it must be the case that for every pair of arming levels $a_i$ and $a'_i$ in a set that occurs with probability one under $F_i$ $U_i(a_i) - U_i(a'_i) = c_i(a_i) - c_i(a'_i)$. Taking limits and applying Theorem 1 we obtain
\[
\int_{m_j}^{\pi_j} \left[ \frac{\partial v_i^D(a_i, a_j)}{\partial a_i} q(a_i, a_j) + \frac{\partial v_i^A(a_i, a_j)}{\partial a_i} (1 - q(a_i, a_j)) \right] dF_j(a_j) = c_i'(a_i).
\]
This means there is a clear relationship between $q$ and $F$ whenever $a_i$ is in the support of player $i$’s mixed strategy. As $v$ and $c$ are exogenous, we conclude that $q$ and $F_j$ have to be “offsetting” in a very specific sense for any two equilibria that use the same investment levels for fixed technologies $v$ and $c$. The next section explores this feature.

### 3.1 Comparative Statics

Suppose we have two equilibria $< t_1, t_2, q, F_1, F_2 >$ and $< \hat{t}_1, \hat{t}_2, \hat{q}, \hat{F}_1, \hat{F}_2 >$ in which investment level $a_i$ is in the supports of both $F_i$ and $\hat{F}_i$. From Theorem 2 we may equate the left hand side of the last equation in both cases and obtain some relationships. In particular, in problems in which the agreement payoff depends just on the transfers (as in the case where investment influence only the outside option) then it is easy to show that the likelihood of disagreement and the marginal value of investing given no settlement are in some sense compliments. That is, if the probability of no settlement given $a_i$ is higher in one equilibrium than the other, then the expected marginal effect from $a_i$ must be ordered in the opposite way across these equilibria. While other statements like this can be obtained in the case where investments influence only the inside options, the comparative statics are most stark in an important special case.

### 3.2 The Case of Additively Separable Disagreement Payoffs

The strength and implications of the relationship in Theorem 2 can be most easily seen if we impose some assumptions on the exogenous functions, $v$. In the analysis of double auctions in which traders possess private information about the agreement payoffs, Fiesler, Kittsteiner and Moldovano (2003) and Kittsteiner (2003) have obtained results for additively separable value functions. In the current context we now show that under this separability condition, i.e $v^A_i(a_i, a_j)$ is of the form $g^A(a_i) + h^A(a_j)$ and

---

4. To make the exposition slightly more natural we write every $a_i$ when we should say something like $F_i$-almost every $a_i$. In the special case of pure strategy equilibria, this system still applies, but reduces to a system of first order conditions.

5. We remind the reader that private values problems are contained in the set of additively separable problems.
\( v_i^D(a_i, a_j) \) is of the form \( g^D(a_i) + h^D(a_j) \) then the probability of no settlement given \( a_i \) is constant. In particular, in the additively separable case, Theorem 2 implies the following result:

**Corollary 1** Fix \( v, c \). If the agreement and disagreement value functions are additively separable then the probability of disagreement given investment level \( a_i \) is the same in every equilibrium in which investment level \( a_i \) is in the support of \( \ell \)'s investment strategy. In particular, the following formula (which has only descriptions of the fundamentals on the right-hand-side) holds

\[
\Pr(\text{disagreement} \mid a_i) = \frac{c^*_i(a_i) - g^{A'}(a_i)}{g^{D'}(a_i) - g^{A}(a_i)}.
\]

In the additively separable case an increase in the marginal cost of investing at level \( a_i \) or a decrease in the marginal value of \( a_i \) from disagreement must result in an increase in the probability of no settlement given \( a_i \). This means that holding fixed technology \( c, v \), variation in the game form \( q, t \) only effects the likelihood of bargaining failure through its effect on the investment strategies. The law of iterative expectations leads to the following corollary.

**Corollary 2** Fix \( v, c \). In the additively separable case, if \( < t_1, t_2, q, F_1, F_2 > \) and \( < t'_1, t'_2, q', F_1, F_2 > \) are equilibria then they both induce the same probability of disagreement. In particular, in this case we have

\[
\Pr(\text{disagreement}) = \int \frac{c^*_i(a_i) - g^{A'}(a_i)}{g^{D'}(a_i) - g^{A}(a_i)} dF_i(a_i)
\]

While the additively separable case is knife-edged, this result illustrates two points. First, an important (and sometimes the only) pathway by which the game form, \( q \) and \( t \), can influence the unconditional probability of bargaining failure is through its influence on the investment strategies. Second, that the pathway by which the choice of game form, \( q \) and \( t \), can influence the conditional probability of bargaining failure depends on the investments being compliments or substitutes (at least locally). We find the effect of these bargaining institutions on conditional probabilities of bargaining failure is continuous in the magnitude of the cross partials of \( v \).
More precisely, define $D_i(a_i) = \min_{a_j} \partial v^P(a_i, a_j)/\partial a_i$, $\overline{D}_i(a_i) = \max_{a_j} \partial v^P(a_i, a_j)/\partial a_i$, $A_i(a_i) = \min_{a_j} \partial v^A(a_i, a_j)/\partial a_i$, and $\overline{A}_i(a_i) = \max_{a_j} \partial v^A(a_i, a_j)/\partial a_i$. Also let

$$\Delta_i(a_i) = \max \{ \overline{D}_i(a_i) - D_i(a_i), \overline{A}_i(a_i) - A_i(a_i) \}$$

and $\Delta_i = \sup_{a_i} \Delta_i(a_i)$. We then have the following result showing continuity in the bounds on the probability of disagreement.

**Theorem 3** Fix $v, c$. If the intervals $[D_i(a_i), \overline{D}_i(a_i)]$ and $[A_i(a_i), \overline{A}_i(a_i)]$ are disjoint then for any $\varepsilon > 0$, there is a $\delta > 0$ s.t. if the primitives, $v, c$ satisfy the condition that $\Delta_i < \delta$ then in any two equilibria $(t_1, t_2, q, F_1, F_2)$ and $(\hat{t}_1, \hat{t}_2, \hat{q}, \hat{F}_1, \hat{F}_2)$ the difference in conditional probability of disagreement, $\left| \int_{a_j} q(a_i, a_j) dF_j(a_j) - \int_{a_j} \hat{q}(a_i, a_j) d\hat{F}_j(a_j) \right| < \varepsilon$ for for every $a_i$ in the supports of both $F_1$ and $\hat{F}_1$.

Thus when payoffs to agreement and disagreement are “close” to additively separable functions—i.e. there are “small” local complementarities—then the difference in the conditional probabilities of disagreement across game forms is small.\(^6\)

### 4 Solutions

Thus far we have focused on necessary conditions that must hold in situations where mixing over investments is optimal given rational expectations about bargaining behavior (i.e., bargaining behavior constitutes an equilibrium given equilibrium conjectures about the investment strategies). In particular, Theorem 2 shows that there is a very specific relationship between the probability of allocations and the investment strategies used in equilibrium. Moreover, this relationship depends on the exogenous primitives $v$ and $c$. In this section we build on Theorem 2 and provide two types of results. The first set of results provide necessary conditions for the types of problems characterized by Theorem 2 to possess solutions and some techniques for finding mixtures when cost functions are given as well as for finding cost functions that can support particular equilibrium descriptions when the remaining primitives $v$ are

\(^6\)The proof of continuity is long, dense, and entirely standard, it is thus relegated to the appendix.
taken as given. The second set of results involves the use of the necessary conditions in Theorem 2 to show that descriptions which are consistent with equilibrium play are fragile. We take a topological approach to the question of whether it is typically the case that we can use the idea of players making hidden investment decisions as a means to support descriptions of bargaining that involve asymmetric information. Specifically we ask when is it the case that their are lotteries over investments that are consistent with equilibrium investment decisions and a particular mechanism, \( q \) for a fixed pair of primitives \( c, v \)? When a solution exists, we say that \( (q, v, c) \) are admissible. We find that even for a nicely behaved class of problems \( (q, v, c) \), the set of inadmissible environments is dense.

Both parts of the analysis of this section involves connecting equation (2) with the theory of integral equations. A technical restriction, makes the characterization more convenient. We focus on the possibility of finding investment strategies with supports contained in a bounded interval.

### 4.1 Investment decisions and Fredholm’s equations of the first kind

We begin investigating the nature of solutions to our investment problems by observing that the equations generated by Theorem 2 are Fredholm integral equations of the first kind. This observation highlights that investment problems are living in the world of solutions to systems of linear equations and the mixing condition induces a set of problems analogous to the problems commonly found in linear functional analysis. To start we note that a Fredholm integral equation of the first kind typically is written in the form

\[
\int_a^b K(x, t)f(t)dt = g(x),
\]

where the function \( K \) is called the kernel and \( f \) and \( g \) are functions of a single variable. In the study of integral equations, and in our problem, \( K \) and \( g \) are taken as given and the item to be solved for is \( f \). From Theorem 2 we have an equilibrium investment
equation that requires for every \(a_i\) in the support of \(F_i\)
\[
\int_{\Omega_j} \left[ \frac{\partial v_i^D(a_i, a_j)}{\partial a_i} q(a_i, a_j) + \frac{\partial v_i^A(a_i, a_j)}{\partial a_i} (1 - q(a_i, a_j)) \right] dF_j(a_j) = c_i'(a_i). \tag{3}
\]

We will say that allocation, comprised of the derivative of \(v_i^A, v_i^D\), and the mechanism \(q\) induce a kernel
\[
K(a_1, a_2) = \left[ \frac{\partial v_i^D(a_1, a_2)}{\partial a_1} q(a_1, a_2) + \frac{\partial v_i^A(a_1, a_2)}{\partial a_1} (1 - q(a_1, a_2)) \right],
\]
and use \(K\) to reduce the notational burden of the analysis. We will also focus on the problem from the perspective of player one, as the considerations (albeit not the exact functions) for player two are the same. To find an investment strategy for a mechanism we are then looking for a function \(f\) that is a solution to the integral equation where the function \(g\) is the derivative of the cost of investing for player one.

Approaching the problem in this way, much like when looking for solutions to linear systems of equations, leads to the basic insight that whether a function \(f\) solving the integral equation can be found depends on properties of the kernel and the cost function \(c_i'\). An interesting and illustrative special case of this problem is one where \(q, v\) generate a “separable” kernel.\(^7\) The kernel of a Fredholm equation is said to be separable if for some finite \(n\), there exists a family of real-valued functions, \(\{\alpha_i(\cdot), \beta_i(\cdot)\}_{i=1}^n\), s.t. \(K(a_1, a_2) = \sum_{i=1}^n \alpha_i(a_1)\beta_i(a_2)\). When the kernel is separable, the integral equation becomes
\[
\sum_{i=1}^n \alpha_i(a_1) \int_{\Omega_2} \beta_i(a_2) f(a_2) da_2 = c_i'(a_1). \tag{4}
\]

The first observation to make is that the terms \(\int \beta_i(a_2) f(a_2) da_2\) do not depend on \(a_1\) and, therefore, a necessary condition for the existence of a solution to a separable Fredholm equation of the first kind is that \(c_i'(a_1)\) is in the space spanned by the \(\alpha_i\)

\(^7\)It is unfortunate for this analysis that the terms “additively separable” and “separable” are used to define two different types of functions. We will always modify separable with the word “additive” where appropriate and reserve separable for the description of kernels. For a nice treatment of separable integral equations see Kanwal (1997).
functions. That is, if the allocation functions \( v \) and mechanism \( q \) can be represented by a separable function, then a necessary condition for there to exist some \( f \) that satisfies player 1’s equilibrium condition is that the cost function is in the span of the functions \( \alpha_i \),

\[
c'_1(a_1) = \sum_{i=1}^{n} \alpha_i(a_1)\gamma_i
\]

(5)

for some set of \( n \) scalars \( \gamma_i \).

Now if we suppose that the cost function is in this subspace, can we find a function \( f \)? Substituting (5) into (4) yields,

\[
\sum_{i=1}^{n} \alpha_i(a_1)\gamma_i = \sum_{i=1}^{n} \alpha_i(a_1) \int_{a_2}^{\pi_2} \beta_i(a_2)f(a_2)da_2
\]

(6)

\[
\sum_{i=1}^{n} \alpha_i(a_1) \left[ \gamma_i - \int_{a_2}^{\pi_2} \beta_i(a_2)f(a_2)da_2 \right] = 0
\]

(7)

A sufficient condition for \( f \) to solve (4) is that the \( f \) solves the \( n \) integral equations,

\[
\gamma_i = \int_{a_2}^{\pi_2} \beta_i(a_2)f(a_2)da_2, \text{ for } i = 1, \ldots, n,
\]

(8)

This system is easier to work with. We can proceed as follows. Suppose that, \( f(\cdot) \) is a linear combination of the functions \( \beta_i(\cdot) \),

\[
f(a_2) = \sum_{i=1}^{n} \beta_i(a_2)b_i
\]

(9)

for \( n \) scalers, \( b_i \). Substituting yields,

\[
\gamma_i = \sum_{j=1}^{n} b_j \int_{a_2}^{\pi_2} \beta_i(a_2)\beta_j(a_2)da_2, \text{ for } i = 1, \ldots, n
\]

\[
\gamma_i = \sum_{j=1}^{n} b_j \langle \beta_i(a_2)\beta_j(a_2) \rangle, \text{ for } i = 1, \ldots, n
\]

(10)
and the unknown variables are the $n$ scalars, $b_j$. This system can be expressed as $Bb = \gamma$ where $B$ is the $n \times n$ symmetric matrix with $i,j$th element $\int \beta_i(a_2)\beta_j(a_2)da_2$ , $b = [b_1, \ldots, b_n]^T$, and $\gamma = [\gamma_1, \ldots, \gamma_n]^T$. We must be careful, however. Even though the Grammian matrix $B$ associated with the set $\beta_j$ is nonsingular, because the $\beta_j$ are independent, and we have a unique $f$ of the form in (9), this is not the only solution to our system. Observe that the system of homogeneous equations

$$\int_{a_2}^{\pi_2} \beta_j(a_2)\xi(a_2)da_2 = 0, \quad j = 1, \ldots, n$$

has more than the trivial solution—because there are a finite number of $\beta_j$—and by linearity of our integral equation there is an entire family of solutions to our original problem

$$\hat{f} = f^* + z\xi,$$

where $f^*$ is the implied solution given $b_i$ from the system of linear equations above, $z$ is a constant, and $\xi$ is a non-trivial solution to the homogeneous integral equation in (11). This is not bad news for us, because we still have to incorporate two additional constraints into our problem: $f$ must be non-negative and, on the support of the investments, it integrates to one.

From the above analysis we can characterize a necessary condition for the existence of a solution in the separable case. With some additional work found in the appendix, a sufficient condition on the primitives of the problem can be given for there to exist a solution that is a density.

**Theorem 4** Suppose that $K(a_1,a_2)$ is a separable kernel induced by $v$ and $q$ with functions $\{\alpha_i(a_1), \beta_i(a_2)\}_{i=1}^n$. Then if a solution exists, the derivative of the cost function is in the span of the set $\{\alpha_i(a_1)\}_{i=1}^n$. 
Moreover let $B'$ be the matrix

$$
B' = \begin{pmatrix}
<\beta_1, \beta_1> & \ldots & <\beta_1, \beta_n> & 0 \\
\vdots & \ddots & \vdots & \vdots \\
<\beta_n, \beta_1> & \ldots & <\beta_n, \beta_n> & 0 \\
<1, \beta_1> & \ldots & <1, \beta_n> & <1, \xi>
\end{pmatrix}
$$

where $<f, g> = \int f(t)g(t)dt$ and $\xi(a_2)$ is a non-trivial solution to the homogeneous integral equation $\int_{a_2}^{a_2} K(a_1, a_2)h(a_2)da_2 = 0$. Then if there is no $y$ in $\mathbb{R}^{n+1}$ such that $y^T \hat{B} \geq 0$ and $y^T \hat{\gamma} < 0$ then for some scalar $z$ there is a density that solves the investment problem of the form

$$
f(a_2) = \sum_{i=1}^{n} \beta_i(a_2)b_i + z\xi(a_2). \quad (12)
$$

Similar intuition as that found in the separable case applies to integral equations that are not separable. The complication comes from finding the right space of functions such that the cost must be in their “span” and deriving the nature of the coefficients relating this orthogonal basis to $\xi'$. 

Suppose we have a problem such that $v, q$ produce a kernel $K$ that is integrable along with $|K|^2$. Now consider two related kernels,

$$
K^*(a_1, a_2) = \int_{a_2}^{a_2} K(s, a_1)K(s, a_2)ds, \quad (13)
$$

$$
K_*(a_1, a_2) = \int_{a_2}^{a_2} K(a_1, s)K(a_2, s)ds. \quad (14)
$$

These two kernels have two relevant properties, each is symmetric in the sense that $K^*(a_1, a_2) = K^*(a_2, a_1)$ and positive in the sense that for real-valued functions $h \in L^2[a, b]$,

$$
\int_{a_1}^{a_1} \int_{a_2}^{a_2} K^*(a_1, a_2)h(a_1)h(a_2)da_1da_2 \geq 0.
$$

Then as shown in Pogorzelski (1966, p.150) the kernels $K^*$ and $K_*$ have only positive...
Next, we construct our set of orthogonal functions by considering the eigenvalues and eigenfunctions of $K^*, K_*$. Pogorzelski (1966, p.158) shows that these two kernels have the same set of eigenvalues defined as constants $\lambda_n^2$ satisfying the property

$$\varphi_n(a_1) = \lambda_n^2 \int_{a_2}^{\bar{a}_2} K^*(a_1, a_2) \varphi_n(a_2) da_2,$$

$$\psi_n(a_1) = \lambda_n^2 \int_{a_2}^{\bar{a}_2} K_*(a_1, a_2) \psi_n(a_2) da_2.$$ 

where $\varphi_n(a_1)$ and $\psi_n(a_1)$ are corresponding eigenfunctions of $\lambda_n^2$. These eigenvalues and functions from $K^*, K_*$ put us in reach of a general necessary condition and algorithm for finding the cost function for a given $f \in L^2[a_2, \bar{a}_2]$ and kernel (induced by $v, q$). The following is a direct application of Schmidt’s Theorem.

**Theorem 5** (Schmidt’s Theorem) Suppose that $K(a_1, a_2)$ and $f(a_2)$ are integrable, together with $|K|^2$, and that

$$\int_{a_2}^{\bar{a}_2} |K(x, y)|^2 dy, \int_{a_2}^{\bar{a}_2} |K(x, y)|^2 dy$$

are both bounded. Then every function $c'_1(a_1)$ having the form

$$c'_1(a_1) = \int_{a_2}^{\bar{a}_2} K(a_1, a_2) f(a_2) da_2$$

(15)

is the sum of its Fourier series with respect to the system $\{\psi_n(a_1)\}$ associated with $K_*(a_1, a_2)$.


That is, if $c'_1$ is the derivative of our cost function, we can re-write it as an infinite sum with coefficients $(c_1)_n = \frac{1}{\lambda_n} \int_{a_2}^{\bar{a}_2} f(a_2) \varphi(a_2) da_2 = \frac{1}{\lambda_n} \langle f(a_2) \varphi(a_2) \rangle$ and

$$c'_1(a_1) = \sum_{n=1}^{\infty} (c_1)_n \psi_n(a_1).$$

(16)
This result is similar in spirit to the result that we found for separable kernels above, where the derivative of the cost function had to be in the span of the \( \alpha_i \) functions in order for any \( f \) to solve the system. When \( f \) is given, we can exactly characterize the coefficients. Also, when \( K \) is separable, the application of Schmidt’s Theorem shows that the constructed functions in the separable case where taken as orthogonal without loss of generality.

**Remark:** Theorem 5 provides a way to take an incentive compatible Bayesian mechanism \((q)\) and lotteries over investments \(f\) and, using equation (16), construct a cost function (actually its derivative) that is consistent with the given problem. Whether the resulting cost function can be reasonably interpreted as a cost function may then be of interest. One might want \( c'_1 \) to be strictly positive for example.

Going in the other direction, we can also describe a condition for the existence of a function \( f \) in the class of \( L^2[\alpha_2, \alpha_2] \) functions that satisfies the integral equation induced by the functions \( q, v, c \). It is an application of Picard’s Theorem (Pogorzelski, 1966).

**Theorem 6 (Picard’s Theorem)** Suppose that \((v, q)\) induces a kernel \( K(a_1, a_2) \) and that there does not exist any function \( h(a_2) \in L^2[\alpha_2, \alpha_2] \) with non-zero norm such that

\[
\int_{\alpha_2}^{\pi_2} K_*(a_1, a_2)h(a_2)da_2 = 0.
\]

Then the integral equation

\[
\int_{\alpha_2}^{\pi_2} K(a_1, a_1)f(a_2)da_2 = c'_1(a_1)
\]

(17)

possess a solution in \( L^2 \) if and only if the series

\[
\sum_{n=1}^{\infty} |\lambda_n(c'_1)_n|^2
\]

converges, where \( \{\lambda_n\} \) is the sequence of eigenvalues of the kernel \( K_*(a_1, a_2) \) and the numbers \( (c'_1)_n \) are the Fourier coefficients of the given function \( c'_1(a_1) \) with respect to the system of eigenfunctions \( \{\psi_n(a_1)\} \) of the kernel \( K_*(a_1, a_2) \).

For our problem we have the immediate corollary.

Corollary 3 If there exists a solution $f^*$ in the set of solutions given by Picard’s Theorem that is everywhere positive and integrates to one, then there is a solution to the investment problem.

So in a similar fashion as the separable case, take an $f$ such that

$$f(a_2) = \sum_{n=1}^{\infty} f_n \varphi_n(a_2).$$

Using equation (16) some manipulations show that an $f$ satisfying the integral equation has coefficients equal to

$$f_n = \lambda_n \langle \ell_1', \psi_n \rangle .$$

Like before, this is not the unique solution. This particular $f$ defines a family of solutions that can be created by adding a constant times any function in the orthogonal complement of the kernel induced by $v$ and $q$. A solution to the investment problem requires that we find such an object that is non-negative for all $a_2$ in the support and that integrates to one. If there does not exist any function $h(a_2) \in L^2[a_2, \bar{a}_2]$ with non-zero norm such that

$$\int_{a_2}^{\bar{a}_2} K_s(a_1, a_2) h(a_2) da_2 = 0,$$

then it can be shown that this solution is unique in $L^2[a_2, \bar{a}_2]$ and we will not expect our investment problem to have a solution that can be interpreted as a distribution function.

4.2 Denseness

The results of the last section provide direction on how one could verify whether any descriptions of equilibrium investment decisions that satisfy the necessary conditions will exist for fixed exogenous primitives (and also suggest how to produce a solution.
when one exists). We now provide some results about the prevalence of problems for which solutions can and cannot exist when the mechanism \( q \) is continuous. This quantification speaks to our earlier claim that modeling the emergence of uncertainty narrows down the set of observations and strategies that are consistent with equilibrium behavior. We will focus only on sufficiently smooth problems for two reasons. First, the analysis is dramatically simpler when we make this choice. Second, our result is in some sense negative, showing that pathological problems are common, as such showing that this is true even when we look only at smooth problems seems to involve minimal loss of interesting generality.

Even though, Picard’s Theorem presents a characterization for a fairly large set of environments, the result about separable kernels will be used to provide a much simpler argument for the denseness of problems which are not admissible in the space of smooth problems. The key to using the result for the separable case is that, since the polynomials are dense in a suitable space of functions, it is sufficient to examine whether problems that can be solved are dense in the space of polynomials. A similar argument can be constructed using the basis that Picard’s Theorem features but this argument relies on mathematics that are less standard in economics with no gain in terms of the strength of the conclusion.

We begin by assuming that the set of feasible investment levels is itself a bounded interval, \([0, b]\). While we do not require that equilibria involve supports that coincide with \([0, b]\), several of the arguments do rely on the assumption that all of the relevant functions are defined on a bounded set. Our primitives live in three topological vector spaces. Let \( \mathcal{P} \) and \( \mathcal{Q} \) each denote the space of continuously differentiable functions mapping \([0, b]^2\) into \(\mathbb{R}^1\). Let \( \mathcal{C} \) denote the space of continuously differentiable functions mapping \([0, b]^1\) into \(\mathbb{R}^1\). We endow each of these three spaces with the sup norm. Thus for any two functions, \( f \) and \( g \), in one of these spaces, the distance between them is \( \sup_x|f(x) - g(x)| \) where the sup is taken

---

8For any particular specification, it is easy to show via dominance arguments that equilibrium investments will be bounded, but here we make the stronger assumption that, independent of the cost functions there is a technological constraint on how much investment is possible.

9All of the following arguments can be adjusted to work if one uses the \( C^1 \) or \( C^2 \) topology, but the notation becomes a bit more cumbersome. That topology would be more desirable if we needed the space to be complete, but this property is inessential to our analysis, so we use the simpler, if slightly less natural topology.
over $x$ in the domain of the relevant function. We use the following notation for
the corresponding norms in these spaces $\|\cdot\|_{\mathcal{P}}, \|\cdot\|_{\mathcal{Q}}, \|\cdot\|_{\mathcal{C}}$. A description is then a
tuple, $\gamma = (v_1^A, v_1^D, v_2^A, v_2^D, q, c_1, c_2)$ with $v_i^d \in \mathcal{P}, q \in \mathcal{Q}$ and $c_i \in \mathcal{C}$. We let
$\Gamma$ denote the product space. We endow the product space with the norm, $\|\gamma\| =\max\{\|v_1^A\|_{\mathcal{P}}, \|v_1^D\|_{\mathcal{P}}, \|v_2^A\|_{\mathcal{P}}, \|v_2^D\|_{\mathcal{P}}, \|q\|_{\mathcal{Q}}, \|c_1\|_{\mathcal{C}}, \|c_2\|_{\mathcal{C}}\}$. We can then metrize the
space of descriptions and as a product space it inherits separability from its com-
ponent spaces, a fact that will be used in the next argument.

Not every tuple, $\gamma = (v, q, c) \in \Gamma$, can be interpreted as an environment in our
case. Several additional constraints must be satisfied: the image of $q$ must
be a subset of $[0,1]$, the payoff functions in $v$ needs to be nondecreasing in their
first argument and nonincreasing in their second argument, and the cost function $c$ must be strictly increasing. We let $E \subset \Gamma$ denote the set of such descriptions.

We will use the following nomenclature: descriptions are ordered tuples of functions;
they live in $\Gamma$. Environments are descriptions satisfying the additional restrictions
described above. Finally, we let $A \subset E$ denote the set of environments for which
it is possible to find a pair of densities $f_1, f_2$ with supports $S_1$ and $S_2$ resulting in
an equilibrium. In other words, if $\gamma \in A$ then there is a pair of densities which
characterize equilibrium investment strategies given $\gamma$ and the mechanism that $\gamma$
specifies is incentive compatible, given the payoffs that $\gamma$ specifies and the beliefs
$f_1, f_2$. We call $A$ the set of admissible environments, and $E - A$ the set of inadmissible
environments.

Theorem 7 The set of inadmissible environments is dense in the set of environ-
ments.

Proof. It is sufficient to show that for any $\epsilon \geq 0$ and any environment $\gamma$ in
the interior of $E$ there is some inadmissible environment $\eta$ with $\|\gamma - \eta\| \leq \epsilon$.$^{10}$ For any
such $\gamma$ and $\epsilon$, the fact that the polynomials are dense in $\mathcal{C}$ implies that we can find
an environment $\mu$ in which every coordinate is a polynomial and that is at most $\epsilon/2$
away from $\gamma$. Since $\gamma$ is assumed to be in the interior of $E$, for $\epsilon$ small enough,
the new description $\eta$ will also be in the interior of $E$. For this $\gamma, \mu, \epsilon$ it remains to

$^{10}$It is not difficult to see that $E$ has a non-empty interior. More precisely it is easy to exhibit
descriptions that involve only polynomials that are in the interior of $E$.\n
26
show that there is another polynomial $\eta$ satisfying two conditions $\|\mu - \eta\| \leq \epsilon/2$ and $\eta \in E - A$. If such a polynomial can be found then by the triangle inequality we will have shown that there is an environment $\eta$ that is inadmissible and no further away from $\gamma$ than $\epsilon$. Again for $\epsilon$ small enough the new description $\eta$ can be in the interior of $E$, since $\mu$ is in the iterior of $E$. From Theorem 4, on seperable integral equations, we see that a necessary condition for a problem with polynomials to be admissible is that the cost function is in the span of the kernal. Building on this fact, observe that if the polynomial $\mu$ is inadmissible then we are done. If $\mu$ is admissible, we can construct $\eta$ in the following way. Let $c_i(\cdot; \eta)$ denote the cost function for player $i$ associated with environment $\eta$. Since $\eta$ is a tuple of polynomials, the cost function $c_i(\cdot; \eta)$ has a finite degree (order of the largest non-zero coefficient), which we denote $n_i$. Simmilarly, let $K_i(\cdot, \cdot; \eta)$ denote the kernal associated with the integral equation characterizing indifference for player $i$. This kernal is also a polynomial and has a finite degree, which we denote $m_i$. We construct $\eta$ as follows. The kernals for $\eta$ are as given by $\mu$ and the cost function for environment $\mu$ are given by adding to $c_i(\cdot; \eta)$ the polynomial having coefficients of 0 on the first $n_i$ terms and then coefficients of $\delta$ on the next $m_i - n_i + 1$ terms and then coefficients of 0 on the remaining terms. This problem is inadmissible as the cost functions under $\eta$ will not be in the span of the kernal associated with $\eta$. We now show that for sufficiently small $\delta \eta$ and $\mu$ are sufficiently close. Given the choice of metric, we have

$$
\|\mu - \eta\| \leq \max \{ \sup_{x \in [a,b]} x^{m_1} \sum_{j=n_1+1}^{m_1+1} \delta x^j, \sup_{x \in [a,b]} x^{m_2} \sum_{j=n_2+1}^{m_2+1} \delta x^j \} \tag{20}
$$

If $b \geq 1$, then the right hand-side of the above is less than

$$
max \{ [m_1 - n_1]b^{m_1}\delta, [m_2 - n_2]b^{m_2}\delta \} \tag{21}
$$

and so selecting $\delta \leq \epsilon[max \{2[m_1 - n_1]b^{m_1}, 2[m_2 - n_2]b^{m_2}\}^{-1}$ we obtain $\|\mu - \eta\| \leq \epsilon/2$. If $b < 1$ a simmilar argument holds. $\blacksquare$

A similar argument can be used to show that environments which induce integral equations possessing solutions are dense in the set of environments. The perturbation involves adding a function to the kernal. The problem with this argument, however,
is that it does not show that admissible environments are dense, because we have no convenient way to guarantee that the solution to an integral equation can be interpreted as a density or distribution function. Given that all of our analysis has focused only in necessary conditions for admissible descriptions there is little payoff, at the moment, to pushing further on general sufficient conditions for the existence of solutions to the integral equations that are distribution functions.

5 Applications

We conclude by considering two applications. In both cases we investigate whether it is possible to found a description of behavior by introducing a stage in which players make investment decisions. In both cases the answer is no, and the brief analysis illustrates an interesting unraveling problem.

Many applications of bargaining theory in economics and politics involve mechanisms \( q \), that are step functions. These “bang-bang” mechanisms are commonly the product of bargaining games with outside options (Muthoo, 1999) or optimal trading mechanism (Myerson and Satterthwaite, 1983) and often separate type profiles into two sets, those where \( q \) is 1 and those where it is 0. When players make investment decisions these discontinuous mechanism are subject to “unraveling problems”. We illustrate this point by considering examples from two distinct applied literatures. The first is the case of double auctions in the setting of bilateral trade with private valuations. The second involves the case of mediated negotiation in the shadow of war with interdependent values. In both cases, we show that some natural, perhaps even canonical descriptions of behavior when types are generated by nature, cannot exist when we assume that the types are themselves endogenous.

Myerson and Satterthwaite (1983) showed the fundamental impossibility of designing a fully efficient Bayesian incentive compatible and individually rational mechanism for negotiating the terms of trade between a buyer and seller with private values. They went on to characterize mechanisms that maximized the expected gains from trade among all the incentive-compatible and individually rational mechanisms. Their Theorem 2, among other things, shows that in the optimal mechanism the probability of trade is monotonic and exhibits a “bang-bang” property, partitioning the set of
type pairs into either trading profiles or non-trading profiles.

Myerson and Satterthwaite show that the linear Bayesian-Nash equilibrium of the $\frac{1}{2}$-double auction characterized by Chatterjee and Samuelson (1983) maximizes the gains from trade over all the games that could be played by traders with uniform-symmetric distributions of values on the unit interval. In this game, the buyer and seller simultaneously propose prices, unaware of the value of the good for their trading partner. If the buyer’s price is higher than the seller’s the object is sold at the average of the two prices; otherwise there is no trade. Chatterjee and Samuelson show that $\sigma_s(v_s) = \frac{2}{3}v_s + \frac{1}{4}$ and $\sigma_b(v_b) = \frac{2}{3}v_b + \frac{1}{12}$ are linear equilibrium strategies in this game for the seller and buyer, respectively.

Now suppose that the valuation of the object is determined through equilibrium behavior in a pre-play investment stage. We can then extend the description of the model to include two strictly increasing costs functions $c_s(a_s)$ and $c_b(a_b)$ and ask: is there any pair of cost functions that lead the buyer and seller to randomize uniformly over $[0, 1]$ and then play the equilibrium that Chatterjee and Samuelson characterized for the case of uniform priors? This investigation is analogous to Theorem 5 and the subsequent remark.

This trading problem is then an easy special case of our environment. Here $v_s^D(a_s, a_b)$ exhibits private values and is thus additively separable and $v_s^A(a_s, a_b) = 0$ identically. For the buyer $v_b^A(a_b, a_s)$ also exhibits private values and is thus additively separable and $v_b^D(a_b, a_s) = 0$. If we consider the linear equilibrium strategies, then we see that trade occurs with probability zero for buyers with a value less than $\frac{1}{4}$. But this means a buyer will never put positive probability on investing in the interval $(0, \frac{1}{4})$ for any strictly increasing cost function and thus the the conjecture that valuations are uniform in an equilibrium unravels. It is then easy to see that any equilibrium where some types trade with probability zero will have the same problem.

It is also the case that the equilibrium conditional probability of trade must be independent of the choice of trading rules, by Corollary 1. In other words, if $a_i$ is in the support of $i’s$ mixed investment strategy in two different equilibria (to the same or different trading rules) then $\Pr(\text{trade} \mid a_i)$ must be the same in both of these equilibria. This finding should be contrasted with the central thrust of the optimal design problem in Myerson and Satterthwaite, where welfare is maximized by
maximizing the probability of trade (while holding fixed the distribution over profiles of valuations). But when the investments are equilibrium reactions to forecasts about the probability of trading, the choice of trading rules cannot influence the probability of trade conditional on investment levels.

We can then conclude that holding the cost functions fixed, in any two equilibria (to the same or different trading rules) with the same mixed investment strategies (by at least one player) the probability of trade must be the same.

Switching topics, in Bester and Warneryd (2006) and Fey and Ramsay (2008), the problem of bargaining in the shadow of war is considered and conditions for the existence of peaceful equilibrium are presented. In these works, the war problem is characterized as a bargaining problem where war is the disagreement payoff and the outcome of war is determined by a type contingent contest function. Both models give a cut off rule, where as a function of the expected payoff to war, if the social loss of war is sufficiently high then there exist mechanisms with equilibria where war occurs with probability zero. For example, in Fey and Ramsay (2008) the condition for the possibility of peaceful equilibrium settlements is that the difference of the expected probability of winning for the strongest type of country one and the expected probability of losing for the strongest type of country two must be less than the sum of the costs of war.

Like the case of bilateral trade, we run into trouble if we are looking for strictly increasing cost functions for arming. Specifically if the countries’ strength is endogenous, then any type that does not fight with positive probability must find it a profitable deviation to secretly not invest or arm, act as if they did, face no risk of war and recoup the costs, $c_i(a_i)$. In fact, with endogenous arming, we have a condition similar to that found in Meirowitz and Sartori (2008). Assume the value of the prize under negotiation is 1. There exists a peaceful (and thus non arming mechanism i.e., $a_1, a_2 = 0$) without external subsidies iff

$$1 \geq \max_{a_1} \{ v_1^D(a_1, 0) - c_1(a_1) \} + \max_{a_2} \{ v_2^D(a_2, 0) - c_2(a_2) \}.$$

The intuition of this result is straightforward. By the above argument, in a peaceful equilibrium $a_i = 0$ and the mechanism must have the probability of war also equal to zero. If $a_1 = 0$ is a best response to $a_2 = 0$ then it must be the case that $t_i(0, 0) \geq$
max\(_m\) \{v^D_i(a_i, 0) - c_i(a_i)\} (to make deviation at arming undesirable) So for both players to be willing to select \(a_1 = a_2 = 0\) the transfers must sum to at least

\[
\max\_{a_1} \{v^D_1(a_1, 0) - c_1(a_1)\} + \max\_{a_2} \{v^D_2(a_2, 0) - c_2(a_2)\}
\]

and thus this transfer is feasible without subsidy only if the inequality holds. Sufficiency follows from the observation that if each of the above transfers is offered a unilateral deviation from \((0, 0)\) is not profitable. If this condition is satisfied then a constant direct mechanism \(t_i(m_i, m_j) = \max\_{a_i} \{v^D_i(a_i, 0) - c_i(a_i)\}\) and \(q(m_i, m_j) = 0\) works. With this mechanism, a deviation in reporting does not improve \(i\)'s payoff unless war occurs, But expression (1) implies that neither player is better off unilaterally deviating and getting war.

This condition is different from the ones in Bester and Warneryd (2006) and Fey and Ramsay (2008). When the probability of war is 0 the equilibrium investment strategies must put probability 1 on the lowest possible investment level. In the above papers war can be avoided if the highest type of each player doesn’t expect to get too high a payoff from fighting against an opponent who’s type is drawn from the prior. Here, the opponent’s type takes on the lowest possible value with probability 1, and this makes the payoff to fighting higher. Accordingly, when the uncertainty is endogenous it is not possible to use uncertainty about the other players capacity to partially deter a player from fighting (along with making a high enough transfer to her). If equilibrium involves peace then equilibrium also involves little doubt about the opponents investment choice.

6 Conclusion

In many applied contexts the choice of bargaining institutions may also influence actions that the agents take in order to influence their agreement and disagreement payoffs. For example when bargaining failure is likely, an agent has a stronger incentive to invest in making her disagreement payoff better. Such a choice can also feed upon itself, as investments in disagreement makes the disagreement payoff more attractive and further influence the incentives of both players while bargaining. As this simple
statement suggests, equilibrium investment decisions and equilibrium bargaining behavior can be linked in important ways. In contrast, much of the literature applying bargaining theory to topics like negotiating in bilateral trade or in the shadow of war posits an exogenous distribution over types and, therefore, payoffs in the game. Extending the canonical models of bargaining to include pre-bargaining investment, we show that aside from capturing an additional feature of some real-life bargaining problems, a focus on pre-bargaining investment offers a theoretical benefit. Imposing equilibrium constraints on investments, and therefore the distributions of payoff values, sometimes limits which distributions over bargaining outcomes are consistent when compared to predictions of Bayesian-Nash equilibria. This increased precision comes from the constraints that equilibrium put on the beliefs held by players in the bargaining game.
A Appendix

A.1 Proof of Theorem 3

Proof. Taking equation 2 in both equilibria and subtracting yields,

\[
\int_{a_i}^{\pi_i} \frac{\partial v_i^D(a_i, a_j)}{\partial a_i} q(a_i, a_j) dF_j(a_j) - \int_{a_i}^{\pi_i} \frac{\partial v_i^P(a_i, a_j)}{\partial a_i} \hat{q}(a_i, a_j) d\hat{F}_j(a_j) = \int_{a_i}^{\pi_i} \frac{\partial v_i^A(a_i, a_j)}{\partial a_i} (1 - q(a_i, a_j)) dF_j(a_j) - \int_{a_i}^{\pi_j} \frac{\partial v_i^A(a_i, a_j)}{\partial a_i} (1 - \hat{q}(a_i, a_j)) d\hat{F}_j(a_j)
\]

(22)

Since \(q, F\) and \(\hat{q}, \hat{F}\) are all non-negative, and the functions \(v^A\) and \(v^D\) are twice differentiable, the mean value theorem for integration implies that for values \(D(a_i), \hat{D}(a_i) \in [\underline{D}(a_i), \overline{D}(a_i)]\) and \(A(a_i), \hat{A}(a_i) \in [\underline{A}(a_i), \overline{A}(a_i)]\) the above equality becomes

\[
D(a_i) \int_{a_i}^{\pi_j} q(a_i, a_j) dF_j(a_j) - \hat{D}(a_i) \int_{a_i}^{\pi_j} \hat{q}(a_i, a_j) d\hat{F}_j(a_j) = \hat{A}(a_i) \int_{a_i}^{\pi_j} (1 - \hat{q}(a_i, a_j)) d\hat{F}_j(a_j) - A(a_i) \int_{a_i}^{\pi_j} (1 - q(a_i, a_j)) dF_j(a_j)
\]

(23)

Since both \(F_j\) and \(\hat{F}_j\) integrate to 1, linearity of the integral yields

\[
(D(a_i) - A(a_i)) \int_{a_i}^{\pi_j} q(a_i, a_j) dF_j(a_j) - (\hat{D}(a_i) - \hat{A}(a_i)) \int_{a_i}^{\pi_j} \hat{q}(a_i, a_j) d\hat{F}_j(a_j) = \hat{A}(a_i) - A(a_i)
\]

(24)

We wish to show that for every \(\varepsilon > 0\) there is some \(\delta > 0\) s.t. \(\Delta_i < \delta\) implies \(\left| \int_{a_i}^{\pi_j} q(a_i, a_j) dF_j(a_j) - \int_{a_i}^{\pi_j} \hat{q}(a_i, a_j) d\hat{F}_j(a_j) \right| < \varepsilon\). Suppose that \(\Delta_i < \delta\). This implies that \(\left| D(a_i) - \hat{D}(a_i) \right| < \delta\) and \(\left| A(a_i) - \hat{A}(a_i) \right| < \delta\) for all \(a_i\). Taking absolute values of both sides of equation (24) yields
\[
(D(a_i) - A(a_i)) \int_{\Omega_j} q(a_i, a_j) dF_j(a_j) - (\hat{D}(a_i) - \hat{A}(a_i)) \int_{\Omega_j} \hat{q}(a_i, a_j) d\hat{F}_j(a_j)
\]
\[
= |\hat{A}(a_i) - A(a_i)| \quad (25)
\]

If \( (D(a_i) - A(a_i)) > (\hat{D}(a_i) - \hat{A}(a_i)) \) then we can rewrite the above as

\[
|\int_{\Omega_j} q(a_i, a_j) dF_j(a_j) - \int_{\Omega_j} \hat{q}(a_i, a_j) d\hat{F}_j(a_j)|
\]
\[
+\left| (D(a_i) - A(a_i)) - (\hat{D}(a_i) - \hat{A}(a_i)) \int_{\Omega_j} \hat{q}(a_i, a_j) d\hat{F}_j(a_j) \right| < \delta \quad (26)
\]

\[
|\int_{\Omega_j} q(a_i, a_j) dF_j(a_j) - \int_{\Omega_j} \hat{q}(a_i, a_j) d\hat{F}_j(a_j)|
\]
\[
+\left| (D(a_i) - \hat{D}(a_i) + \hat{A}(a_i) - A(a_i)) \int_{\Omega_j} \hat{q}(a_i, a_j) d\hat{F}_j(a_j) \right| < \delta \quad (27)
\]

Since \( (D(a_i) - A(a_i)) - (\hat{D}(a_i) - \hat{A}(a_i)) \leq \delta \) the left hand side is bigger than

\[
|\int_{\Omega_j} q(a_i, a_j) dF_j(a_j) - \int_{\Omega_j} \hat{q}(a_i, a_j) d\hat{F}_j(a_j)|,
\]

or it is bigger than

\[
|D(a_i) - A(a_i)| \left[ \int_{\Omega_j} q(a_i, a_j) dF_j(a_j) - \int_{\Omega_j} \hat{q}(a_i, a_j) d\hat{F}_j(a_j) \right] - 2\delta \int_{\Omega_j} \hat{q}(a_i, a_j) d\hat{F}_j(a_j).
\]

In the first case this implies that

\[
|D(a_i) - A(a_i)| \left| \int_{\Omega_j} q(a_i, a_j) dF_j(a_j) - \int_{\Omega_j} \hat{q}(a_i, a_j) d\hat{F}_j(a_j) \right| < \delta,
\]

so that

34
\[ \int_{a_j}^{\pi_j} q(a_i, a_j) dF_j(a_j) - \int_{a_j}^{\pi_j} \hat{q}(a_i, a_j) d\hat{F}_j(a_j) < \frac{\delta}{|(D(a_i) - A(a_i))|}, \]

which implies that as long as \([D_i(a_i), D_i(a_i)]\) and \([A_i(a_i), A_i(a_i)]\) are disjoint (so that \(|(D(a_i) - A(a_i))| > 0\)), for every \(\varepsilon > 0\), if \(\delta < \varepsilon \Delta_i\) then

\[ \int_{a_j}^{\pi_j} q(a_i, a_j) dF_j(a_j) - \int_{a_j}^{\pi_j} \hat{q}(a_i, a_j) d\hat{F}_j(a_j) < \varepsilon. \]

In the second case we obtain

\[ |(D(a_i) - A(a_i))| \int_{a_j}^{\pi_j} q(a_i, a_j) dF_j(a_j) - \int_{a_j}^{\pi_j} \hat{q}(a_i, a_j) d\hat{F}_j(a_j) < 2\delta \int_{a_j}^{\pi_j} \hat{q}(a_i, a_j) d\hat{F}_j(a_j) < \delta. \]

And since \(\int_{a_j}^{\pi_j} \hat{q}(a_i, a_j) d\hat{F}_j(a_j) < 1\), we obtain

\[ \int_{a_j}^{\pi_j} q(a_i, a_j) dF_j(a_j) - \int_{a_j}^{\pi_j} \hat{q}(a_i, a_j) d\hat{F}_j(a_j) < \frac{3\delta}{|(D(a_i) - A(a_i))|}, \]

and the previous conclusion holds with the bound \(\delta < \frac{\varepsilon}{3} \Delta_i\).

\[ \blacksquare \]

### A.2 Proof of the sufficiency in Theorem 4

**Proof.** Suppose that \(v, q\) induce a separable kernel \(K(a_1, a_2)\) represented by a set of functions \(\{\alpha_i(a_1), \beta_i(a_2)\}_{i=1}^n\). As our problem always produces positive kernels, without loss of generality, we can assume the \(\beta_i(a_2)\) are non-negative. Following the steps in Section 4.1, we know that if we were only concerned with finding a function \(f\) to solved the integral equation we would end up with the Grammian matrix \(B\), having rank \(n\), and a family of solutions

\[ f(a_2) = \sum \beta_i(a_2) b_i + z \xi(a_2), \]
for any $\xi(a_2)$ in the orthogonal complement of $K(a_1, a_2)$. We now augment this problem by restricting the solution to integrate to one. That is, we want to find an $f(a_2)$ such that

$$
\int f(a_2) da_2 = \int_{a_2} \left[ \sum \beta_i(a_2) b_i + z \xi(a_2) \right] da_2 = \\
\sum b_i \int_{a_2} \beta_i(a_2) da_2 + z \int_{a_2} \xi(a_2) da_2 = 1.
$$

Recalculating the above analysis, and recalling that because $\xi(a_2)$ is in the orthogonal complement of our problem, we end up with a new linear system of equations $B'\hat{b} = \gamma$,

$$
\begin{pmatrix}
<\beta_1, \beta_1> & \ldots & <\beta_1, \beta_n> & 0 \\
\vdots & \ddots & \vdots & \vdots \\
<\beta_n, \beta_1> & \ldots & <\beta_n, \beta_n> & 0 \\
<1, \beta_1> & \ldots & <1, \beta_n> & <1, \xi>
\end{pmatrix}
\begin{pmatrix}
b_1 \\
\vdots \\
b_n \\
z
\end{pmatrix}
= 
\begin{pmatrix}
\gamma_1 \\
\vdots \\
\gamma_n \\
1
\end{pmatrix}
$$

So $B'$ is a matrix and $\hat{b}$ is a vector of weights that make $f$ integrate to one and solve the integral equation for investments. As $B$ is a Grammian matrix, and the $\beta_i$ are linearly independent, it has a non-zero determinant and is non-singular. Then, because $\xi(a_2)$ is a non-trivial solution to the homogeneous integral equation, $B'$ is also non-singular and generates a unique solution $\hat{b}$.

Finally, recall that the set of functions $\{\beta_i(a_2)\}_{i=1}^n$ are positive for all $i$ and $a_2$. It is then straightforward that a sufficient condition for $f(a_2)$ to be a density is that the resulting $b_i$ are positive for all $i$. If there is no $y$ in $R^{n+1}$ such that $y^T B' \geq 0$ and $y^T \hat{\gamma} < 0$ then by Farkas’s Lemma there exists a solution with $\hat{b} \geq 0$, proving the result. ■
References


